Conserved Quantities and the Evolution of Perturbations in Lemaître-Tolman-Bondi Cosmology

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Overview

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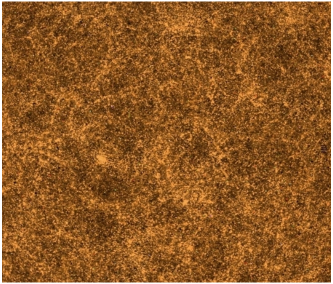
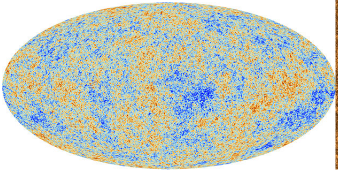
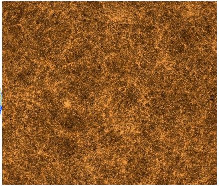


Image: SDSSIII

Why Perturb LTB Cosmology?





Homogeneous versus Inhomogeneous Cosmologies and Inflation

- **What if the universe is not homogeneous on large scales?**
- **What if the universe is inhomogeneous on -some- larger scale?**
- **What if the growth of structure in the universe is governed by perturbations around an inhomogeneous background cosmology?**
- **Simplest of these to investigate: Lemaître-Tolman-Bondi Cosmology**

Flat FRW vs LTB

Flat FRW vs LTB

- FRW: Maximally symmetric spatial section - expansion time dependent only

$$ds^2 = -dt^2 + a^2(t)dr^2 + a^2(t)r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- LTB: Spherically symmetric spatial section - expansion time and r coordinate dependant (not θ, ϕ)[a]

$$ds^2 = -dt^2 + X^2(r,t)dr^2 + Y^2(r,t)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

[a] Bondi 1947

The Standard Model of Cosmology - Flat FRW

The Standard Model of Cosmology - Flat FRW

- Background metric:

$$ds^2 = -dt^2 + a^2\delta_{ij}dx^i dx^j$$

- Perturbed metric:

$$ds^2 = -(1+2\Phi)dt^2 + 2aB_i dx^i dt + a^2(\delta_{ij} + 2C_{ij})dx^i dx^j$$

with scalar, vector and tensor perturbations[a]

[a] e.g. Bardeen 1980

The Standard Model of Cosmology - Flat FRW

The Standard Model of Cosmology - Flat FRW

- Further decomposition of 3-spatial perturbations gives curvature perturbation ψ, identified with the intrinsic scalar curvature:

$$C_{ij} = E_{,ij} - \psi\delta_{ij} + vector + tensor \quad quantities^*$$

* On 3-spatial hypersurfaces

The Standard Model of Cosmology - Flat FRW

Constructing Gauge Invariant Quantities

- Splitting quantities into background + perturbation: no longer covariant - gauge dependent; construct gauge invariant quantities
- General gauge transformations:

$$\widetilde{\delta \mathbf{T}} = \delta \mathbf{T} + \pounds_{\delta x^\mu} \bar{\mathbf{T}}$$

- Tilde denotes new coordinates

$$\widetilde{x^\mu} = x^\mu + \delta x^\mu$$

  bar denotes background.
- Key quantities gauge transformations:

$$\begin{aligned} \widetilde{\psi_{\rm FRW}} &= \psi_{\rm FRW} + \frac{\dot{a}}{a}\delta t \\ \widetilde{\delta\rho_{\rm FRW}} &= \delta\rho_{\rm FRW} + \dot{\bar{\rho}}\delta t \end{aligned}$$

The Standard Model of Cosmology - Flat FRW

Constructing Gauge Invariant Quantities

- Gauge choice: uniform density hypersurfaces, $\widetilde{\delta\rho_{\rm FRW}} = 0$

$$\delta t = -\frac{\delta\rho_{\rm FRW}}{\dot{\bar{\rho}}}$$

Get gauge invariant curvature perturbation

$$-\zeta \equiv \psi_{\rm FRW} + \frac{\dot{a}/a}{\dot{\bar{\rho}}}\delta\rho$$

- Evolution equations for ζ from time derivative, $\delta\rho$ from energy conservation $\nabla_\mu T^{\mu\nu} = 0$...

  ζ conserved in large scale limit - conserved perturbed quantities allow easily relate early to late times (e.g. curvature/physics early time relates to density/observables late time)

Conserved Quantities in Perturbed LTB

Perturbed LTB

- Background metric:

$$ds^2 = -dt^2 + X^2(r,t)dr^2 + Y^2(r,t)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- Perturbed Metric:

$$ds^2 = -(1+2\Phi)dt^2 + 2B_i d\mathcal{X}^i dt + (\delta_{ij} + 2C_{ij})d\mathcal{X}^i d\mathcal{X}^j$$

  where $d\mathcal{X}^i = [X(r,t)dr, Y(r,t)d\theta, Y(r,t)\sin\theta d\phi]$ and we reserve dx^i for $[dr, d\theta, d\phi]$

-

$$X(r,t) = \frac{1}{W(r)}\frac{\partial Y(r,t)}{\partial r},$$

  where $W(r)$ is an arbitrary function of r.

Conserved Quantities in Perturbed LTB

Perturbed LTB

- We have performed 1+3 decomposition into time and spatial sections of metric
- In FRW we have S, V, T decomposition, here it is even more complicated and needs spherical harmonics but...[a]
- our undecomposed perturbations give simpler expressions, easing constructing conserved quantities.

[a] e.g. Clarkson, Clifton, February 2009, Gundlach, Martin-Garcia 2000, Gerlach, Sengupta 1979

Conserved Quantities in Perturbed LTB

LTB Governing Equations

- Background Energy Conservation:

$$\dot{\rho} + \rho(H_X + 2H_Y) = 0\,, \qquad H_X = \frac{\dot{X}}{X}\,, \qquad H_Y = \frac{\dot{Y}}{Y}$$

- Perturbed Energy Conservation:

$$\begin{aligned} \delta\dot{\rho} &+ (\delta\rho + \delta P)\,(H_X + 2H_Y) + \bar{\rho}' v^r \\ &+ \bar{\rho}\Big(\dot{\boldsymbol{C}}_{\boldsymbol{rr}} + \dot{\boldsymbol{C}}_{\boldsymbol{\theta\theta}} + \dot{\boldsymbol{C}}_{\boldsymbol{\phi\phi}} + \partial_r v^r + \partial_\theta v^\theta + \partial_\phi v^\phi \\ &+ \left[\frac{X'}{X} + 2\frac{Y'}{Y}\right] v^r + \cot\theta v^\theta\Big) = 0 \end{aligned}$$

- Convenient to define spatial metric perturbation:

$$3\psi_{\mathrm{SMTP}} = \frac{1}{2}\delta g^k_k = \boldsymbol{C}_{\boldsymbol{rr}} + \boldsymbol{C}_{\boldsymbol{\theta\theta}} + \boldsymbol{C}_{\boldsymbol{\phi\phi}}$$

Conserved Quantities in Perturbed LTB

Constructing Gauge Invariant Quantities

- $\psi_{\rm SMTP}$ transformation behaviour:
  $$3\tilde{\psi}_{\rm SMTP} = 3\psi_{\rm SMTP} + \left[\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right]\delta t + \left[\frac{X'}{X} + 2\frac{Y'}{Y}\right]\delta r + \partial_i \delta x^i + \delta\theta\cot\theta\,,$$
- $\delta\rho$ and 3-velocity transformation behaviour:
  $$\delta\tilde{\rho} = \delta\rho + \dot{\bar{\rho}}\delta t + \bar{\rho}'\delta r$$
  $$\tilde{v}^i = v^i - \delta\dot{x}^i$$
- Gauge choices; uniform density (partially fixes the t coordinate):
  $$\delta t\Big|_{\delta\tilde{\rho}=0} = -\frac{1}{\dot{\bar{\rho}}}\left[\delta\rho + \bar{\rho}'\delta r\right]$$
  comoving (completes the gauge fixing in the spatial coordinates):
  $$\delta x^i = \int v^i dt$$

Conserved Quantities in Perturbed LTB

Constructing Gauge Invariant Quantities

- Gives gauge invariant **Spatial Metric Trace Perturbation (SMTP)** on comoving, uniform density hypersurfaces:

$$
\begin{aligned}
-\zeta_{\mathbf{SMTP}} &= \psi_{\mathrm{SMTP}} + \frac{\delta\rho}{3\bar{\rho}} + \frac{1}{3}\Bigg\{ \left(\frac{X'}{X} + 2\frac{Y'}{Y} + \frac{\bar{\rho}'}{\bar{\rho}}\right)\int v^r dt \\
&+ \partial_r \int v^r dt + \partial_\theta \int v^\theta dt + \partial_\phi \int v^\phi dt + \cot\theta \int v^\theta dt \Bigg\}
\end{aligned}
$$

Conserved Quantities in Perturbed LTB

Constructing Gauge Invariant Quantities

- Gauge invariant density perturbation on uniform curvature hypersurfaces also constructed (i.e. $\widetilde{\psi}_{\rm SMTP} \equiv 0$)

$$\delta t = -\frac{1}{H_X + 2H_Y}\left[3\psi_{\rm SMTP} + \left(\frac{X'}{X} + 2\frac{Y'}{Y}\right)\delta r + \partial_i \delta x^i + \delta\theta \cot\theta\right]$$

$$\delta\tilde{\rho}\Big|_{\widetilde{\psi}_{\rm SMTP}=0} = \delta\rho + \bar{\rho}\left\{3\psi + \left(\frac{X'}{X} + 2\frac{Y'}{Y} + \frac{\bar{\rho}'}{\bar{\rho}}\right)\int v^r\,dt \right.$$

$$\left. + \quad \partial_r \int v^r\,dt + \partial_\theta \int v^\theta\,dt + \partial_\phi \int v^\phi\,dt + \cot\theta \int v^\theta\,dt\right\}$$

- May be related to $\zeta_{\rm SMTP}$ as

$$\delta\tilde{\rho}\Big|_{\widetilde{\psi}_{\rm SMTP}=0} = -3\bar{\rho}\zeta_{\rm SMTP}$$

- c.f. ζ and $\delta\tilde{\rho}\Big|_{\widetilde{\psi_{\rm FRW}}=0}$ in flat FRW

Conserved Quantities in Perturbed LTB

Conserved Quantities in Perturbed LTB

- $\zeta_{\rm SMTP}$ Evolution Equation:

$$\dot{\zeta}_{\mathbf{SMTP}} = \frac{H_X + 2H_Y}{3\bar{\rho}} \delta P_{\mathbf{nad}}$$

- Valid on all scales.
- For barotropic fluids $\dot{\zeta}_{\mathbf{SMTP}} = 0$

Conserved Quantities in Perturbed LTB

Conclusion and Further Research

- $$\dot{\zeta}_{\rm SMTP} = \frac{H_X + 2H_Y}{3\bar{\rho}}\delta P_{\rm nad}$$

- Research already extended to other spacetimes
  i.e. $\dot{\zeta}_{\rm SMTP}$ already extended to Lemaitre and FRW

- $\zeta_{\rm SMTP}$ can provide a useful analytical check for numerical codes in perturbed LTB/inhomogeneous spacetimes...
  as described in 1412.3012 Bartelmann et al.

- e.g. potentially useful in study of structure formation in large voids (see e.g. 1411.1828 Das et al.)

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Additional notes - time permitting

Additional Notes

- Comparison with 2+2 Spherical Harmonic Decomposition
- ζ in Clifton, Clarkson, February formalism:

$$
\begin{aligned}
-\zeta_{\mathrm{SMTP}} \quad = \quad & \frac{1}{6}\left(\frac{(1-\kappa r^2)}{a^2{}_{\parallel}} h_{rr}\mathcal{Y} + \frac{h\ \bar{\mathcal{Y}}_{\theta\theta}}{a_{\perp}{}^2 r^2} + \frac{h\ \bar{\mathcal{Y}}_{\phi\phi}}{a_{\perp}{}^2 r^2 \sin^2\theta} + 2K\mathcal{Y} + G\mathcal{Y}_{:\theta\theta} + \frac{G\mathcal{Y}_{:\phi\phi}}{\sin^2\theta}\right) \\
+ \quad & \frac{\delta\rho}{3\bar{\rho}} + \frac{1}{3}\Bigg\{\partial_\theta \int \frac{1}{a_{\perp}{}^2 r^2}\left(\bar{\mathrm{v}}\ \bar{\mathcal{Y}}_\theta + \tilde{\mathrm{v}}\ \mathcal{Y}_\theta - h_t^{\mathrm{axial}}\bar{\mathcal{Y}}_\theta - h_t^{\mathrm{polar}}\mathcal{Y}_\theta\right) dt \\
+ \quad & \partial_\phi \int \frac{1}{a_{\perp}{}^2 r^2 \sin^2\theta}\left(\bar{\mathrm{v}}\ \bar{\mathcal{Y}}_\phi + \tilde{\mathrm{v}}\ \mathcal{Y}_\phi - h_t^{\mathrm{axial}}\bar{\mathcal{Y}}_\phi - h_t^{\mathrm{polar}}\mathcal{Y}_\phi\right) dt \\
+ \quad & \cot\theta \int \frac{1}{a_{\perp}{}^2 r^2}\left(\bar{\mathrm{v}}\ \bar{\mathcal{Y}}_\theta + \tilde{\mathrm{v}}\ \mathcal{Y}_\theta - h_t^{\mathrm{axial}}\bar{\mathcal{Y}}_\theta - h_t^{\mathrm{polar}}\mathcal{Y}_\theta\right) dt \\
- \quad & \partial_r \int \frac{\mathcal{Y}(1-\kappa r^2)}{a_{\parallel}{}^2}\left(\frac{1}{2}h_{tr} + \frac{a_{\parallel}}{\sqrt{(1-\kappa r^2)}}\tilde{w}\right) dt \\
- \quad & \left[\left(\frac{a_{\parallel}}{\sqrt{(1-\kappa r^2)}}\right)^{\dagger} + 2\frac{(a_{\perp} r)^{\dagger} a_{\parallel}}{a_{\perp} r\sqrt{(1-\kappa r^2)}}\right. \\
+ \quad & \left.\frac{\bar{\rho}^{\dagger} a_{\parallel}}{\bar{\rho}\sqrt{(1-\kappa r^2)}}\right] \int \frac{(1-\kappa r^2)}{a^2{}_{\parallel}}\mathcal{Y}\left(\frac{1}{2}h_{tr} + \frac{a_{\parallel}}{\sqrt{(1-\kappa r^2)}}\tilde{w}\right) dt\Bigg\},
\end{aligned}
$$