Compressible Hall magnetohydrodynamics in a strong magnetic field

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Plasma dynamics becomes anisotropic in the presence of a strong background magnetic field, a feature that may be exploited to yield reduced fluid models. The reduced Hall-magnetohydrodynamics model derived in a recent work by Gomez et al. [Phys. Plasmas 15, 102303 (2008)], reflecting two-fluid effects such as the Hall current and the electron pressure, is extended to account for a crucial aspect of the role of the plasma compressibility, i.e., the compression of the guide field. This reduced model constitutes therefore a description of the two-fluid plasma dynamics in a strong external magnetic field, which can be used also for values of the plasma pressure parameter $\beta$ of the order of unity or smaller. © 2009 American Institute of Physics. [DOI: 10.1063/1.3159862]

Fluid models are valuable frameworks for the understanding of the physics of magnetized plasmas. One-fluid magnetohydrodynamics (MHD) is a standard description of the large-scale dynamics of plasmas, which fails however to describe fluid phenomena with characteristic length scales smaller than the ion skin depth, $d_i = c/\omega_{pi}$, with $\omega_{pi}$ as the ion plasma frequency. A well-known reason is that for scales below $d_i$, it becomes inaccurate to make the approximation that the magnetic field is carried by the bulk flow of ions, the latter being instead frozen-in the electron component of the plasma flow. Therefore, two-fluid effects become important at scales smaller than $d_i$, and it is necessary to refine the description of the electron component of the fluid, as it is done for instance in Hall MHD, which extends the validity domain of MHD and accounts for the Hall current and electron pressure effects.

In many situations of interest it is legitimate to consider that the plasma is embedded in a strong and regular magnetic field $B_0$. When the background field is only slightly perturbed by the plasma motions, i.e., $\delta B/B_0 \ll 1$ and $\delta V/V_A \ll 1$, with $V_A$ being the Alfvén velocity, an important consequence is that the plasma tends to develop length scales that are much smaller in the direction perpendicular to $B_0$ than they are in the direction parallel to it, i.e., $k_j/k_i \ll 1$. This anisotropy of the plasma dynamics can be exploited to yield reduced fluid models. The reduced-MHD (RMHD) model, first introduced in Refs. 1 and 2, can rigorously be obtained from an asymptotic expansion of the compressible MHD equations under the ordering $k_i/k_j \sim \delta B/B_0 \sim \delta V/V_A \ll 1$. In the reduction scheme, the fast time scale is the propagation time of the compressional Alfvén mode, which is therefore eliminated, while the low-frequency dynamics of the shear Alfvén and slow modes are retained, the slow mode becoming a sound wave in a low-$\beta$ plasma. Within RMHD, the nonlinear shear Alfvénic disturbances evolve independently, being described by a stream and a flux function from which the perpendicular perturbations $\delta V_\perp$ and $\delta B_\perp$ may be derived, the parallel perturbations $\delta B_\parallel$ and $\delta V_\parallel$ playing only a passive role. The self-consistency of the RMHD approximation has been investigated and its validity, in the turbulent regime, has been tested numerically by directly comparing its predictions with the ones of the compressible MHD equations.

In a recent work, a system of reduced Hall-MHD (RHMD) equations was derived from the incompressible Hall MHD following the same asymptotic procedure, which is employed to obtain the conventional RMHD from MHD (see Ref. 4 for more references concerning RMHD, its various derivations, and applications). A main limitation of this RHMD model is that because incompressibility of the plasma flow is assumed, the model is only valid for values of the plasma pressure parameter $\beta$, which are much larger than unity, leaving the Hall effect as the sole mechanism for the production of parallel perturbations in the background magnetic field. Although within the RMHD ordering the plasma compressibility remains negligible in the dynamics of the perpendicular perturbations, leading to the decoupling of the shear Alfvén polarized perturbations, compressibility can make important contribution to the dynamics of parallel perturbations. The reason is that the effect of the small compression of the large guide field produces a sizeable $\delta B_\parallel$ and therefore cannot be neglected in a plasma with pressure parameter $\beta$ of the order of unity or smaller. The aim of this communication is to extend the RHMD model derived in Ref. 4 to also account for this effect of the plasma compressibility in the two-fluid description of plasmas embedded in a strong magnetic field.

Under the assumption of quasineutrality, $n_e = n_i = n$, with $n$ as the plasma number density and considering that the electrons have negligibly small inertia, the fluid equations of motion for the ions and the electrons are

$$n m_i (\partial_t V_i + V_i \cdot \nabla V_i) = - \nabla P_i + ne (E + V_i \times B),$$

$$0 = - \nabla P_e - ne (E + V_e \times B),$$

where $V_{ie} = V_i - V_e$ is the ion/electron velocity, $m_i$ is the ion mass, $P_{ie}$ the ion/electron pressure, $E$ is the electric field, and $B$ is the magnetic field. Recalling that the current density $j$ is defined as $j = ne (V_i - V_e)$, the system is supplemented by Maxwell’s equations: $\nabla \times E = - \partial_t B$, $\nabla \times B = \mu_0 j$, and $\nabla \times B = 0$. These
equations are made dimensionless by introducing a typical length scale $L_0$, density $n_0$, a typical value for the magnetic field $B_0$, corresponding to the Alfvén velocity $V_A = B_0 / \sqrt{(\mu_0 n_0)}$, a time scale $L_0 / V_A$, and the pressures are normalized to the magnetic pressure $B_0^2 / \mu_0$.

Equations (1) and (2) are then combined to give an ion equation of motion,

$$\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla P + \mathbf{j} \times \mathbf{B},$$

with $P = P_i + P_e$, while the electron equation of motion is equivalent to the generalized Ohm’s law,

$$\mathbf{E} + \mathbf{V}_e \times \mathbf{B} = -\partial_t \mathbf{V}_e,$$

with $\mathbf{V}_e = \mathbf{V} - d \int \mathbf{V}$. (V = $\mathbf{V}$).

Equation (4) is the statement of the frozen-in condition of the magnetic field in the flow of electrons and not in the bulk flow of ions as it is assumed in the case of standard MHD. This difference of two-fluid MHD with standard MHD involves the additional non-dimensional parameter $d_z$, which is the normalized ion skin depth $d_z = (c/\omega_p) / L_0$ with $\omega_p = \sqrt{(nne^2/ee_m)}$. Assuming the plasma flow to be incompressible $\nabla \cdot \mathbf{V} = 0$ and a uniform background density, Eq. (3) and the curl of Eq. (4), i.e.,

$$\partial_t \mathbf{B} = \nabla \times [(\mathbf{V} - d \mathbf{j}) \times \mathbf{B}] = 0,$$

constitute the set of incompressible Hall-MHD equations. Hall MHD extends the validity domain of MHD to (normalized) length scales of the order of $d_z$ or smaller provided these scales are also larger than $d_z = (c/\omega_p) / L_0 = \sqrt{(nne^2/ee_m)}$, otherwise, the effect of the electron inertia becomes important. Taking V = 0 in the previous equations, i.e., assuming the ions to be at rest, defines the electron MHD.

The existence of a strong background magnetic field $B_B$ makes the plasma dynamics anisotropic with $e = e_z / e_\parallel \ll 1$, i.e., structures have a size much smaller in the direction perpendicular than parallel to the background field. We can write the normalized magnetic field as $B = z + \mathbf{B}$ and make the following ordering $B_B = e$ for its perturbation. The solenoidal condition for the magnetic field perturbation provides that its perpendicular component can be written in term of a flux function: $\mathbf{B} = \nabla \psi \times z + \mathbf{B}$. In the same way, the perpendicular velocity is written in terms of a stream function, $\mathbf{V} = \nabla \psi \times z + u_z \mathbf{z}$, with the ordering $\mathbf{V} \sim \mathbf{B}$. Following the same standard procedure employed to obtain RMHD from the MHD equations, the Hall-MHD equations (3)–(5) yield, at the order $e^2$ in the asymptotic expansion, to

$$\partial_t \psi = \partial_z \phi - dB_z, [\phi - dB_z, \psi],$$

$$\partial_t u_z = \partial_z (v_z - dB_z) + [v_z - dB_z, \psi] + [\phi, b_z] - \nabla \cdot \mathbf{V},$$

$$\partial_t \omega_z = \partial_z j_z + [j_z, \psi] + [\phi, \omega_z],$$

$$\partial_t v_z = \partial_z b_z + [b_z, \psi] + [\phi, v_z],$$

where the notation $[A, B] = \nabla (A \times \nabla B)$ is adopted and $j_z = -\nabla^2 \psi$, $\omega_z = -\nabla^2 \phi$ are, respectively, the $z$-component of the current and of the vorticity. The last compressibility term must be retained in Eq. (7) because it involves $\nabla \cdot \mathbf{V}$, which is of the order of $e^2$, as can be understood by considering the density equation under the ordering $n \sim e$ for the normalized density fluctuation.

The system can be closed by the pressure equation,

$$\partial_t \rho = [\phi, \rho] - \beta \nabla \cdot \mathbf{V},$$

with the ordering, $\rho \sim e$, for the normalized pressure perturbation. The plasma pressure parameter is defined as $\beta = C_s^2 / V_A^2 = \Gamma P_0 / (B_0^2 / \mu_0)$, with $\Gamma$ as the ratio of specific heats and $P_0$ as the background reference pressure. Let us note that in Eqs. (6)–(10) it is possible to introduce the convenient notations, $\nabla A = \partial A + [A, \psi]$, for the derivative along the field lines and $dA = \partial A + [A, \beta]$ for the convective derivative. In deriving Eqs. (6)–(10), we adopted the same conventions and approximations as in Ref. 5 without the restriction to two-dimensional perturbations.

In the limit $\beta \gg 1$ it then follows from Eq. (10) that the plasma flow becomes incompressible with $\nabla \cdot \mathbf{V} = 0$ and the above system is equivalent to the incompressible reduced-Hall MHD equations derived by Gomez et al.4 from the expansion of Eqs. (3) and (5) at the order of $e^2$. As the original Hall-MHD system, this reduced system conserves the total, kinetic plus magnetic energy

$$E = \int d^3 r (|\nabla \psi|^2 + v_z^2 + (\nabla \psi)^2 + b_z^2).$$

For $d_z = 0$, the incompressible RMHD equations are recovered with Eqs. (6) and (8) forming an independent system. It describes the nonlinear dynamics of shear Alfvén wave polarized perturbations, which is decoupled from the dynamics of the slow wave polarized perturbations and, therefore, the energies $E_1 = \int d^3 r (|\nabla \psi|^2 + (\nabla \psi)^2)$ and $E_2 = \int d^3 r (|v_z|^2 + b_z^2)$ can be considered as separate conserved quantities.

Linearizing Eqs. (6)–(9) with $\nabla \cdot \mathbf{V} = 0$ and considering that all the fields vary like exp($ik_z \cdot r + ik_z \cdot \mathbf{z})$, the following dispersion relation is obtained:

$$\omega^2 - 2k_z^2 [1 + \frac{1}{2} (k_z d_z)^2] \omega^2 + k_z^4 = 0.$$ (12)

In the limit $k_z d_z \gg 1$, we have

$$\omega^2 = \frac{k_z^2 d_z^2}{2} \left[ \left( 1 - \frac{4}{k_z^4 d_z^4} \right)^{1/2} \right],$$ (13)

which corresponds to the dispersion relation of the whistler waves with

$$\omega = \pm k_z d_z k_z$$ (14)

and to the dispersion relation of the ion-cyclotron waves with

$$\omega = \pm k_z (k_z d_z).$$ (15)

Taking $\mathbf{V} = 0$ in the previous RMHMD model leads to the reduced electron-MHD equations,

$$\partial_t \psi = -d \partial_z b_z - d [b_z, \psi],$$ (16)

$$\partial_t b_z = -d \partial_z j_z - d [j_z, \psi].$$ (17)

This two-field model conserves the magnetic energy and its linear modes are the whistler waves with $\omega \sim \pm k_z d_z k_z$. Moreover, we notice that $b_z$ plays the role of a stream func-
tion in the dynamics of $\psi$, $\phi_d b_z$ being the generalized stream function in Eq. (6).

Our main goal here is to relax the assumption of a large $\beta$ allowing for the effect of a finite plasma compressibility. Since the perpendicular pressure balance equation, $\nabla_z (p + b_z) = 0$, is satisfied at the order of $\epsilon$ in the expansion of the ion equation of motion, it follows that $\nabla \cdot \mathbf{V}$ can be eliminated from Eqs. (10) and (7) using the fact that $p = b_z$.

$$\partial_t \psi = \partial_z (\phi - d_b Z) + [\phi - d_b Z, \psi], \quad (18)$$

$$\partial_t b_z = \left( \frac{\beta}{\beta + 1} \right) \partial_z (v_z - d_j z) + [v_z - d_j z, \psi] + [\phi, b_z], \quad (19)$$

$$\partial_t \omega_z = \partial_z j_z + [j_z, \psi] + [\phi, \omega_z], \quad (20)$$

$$\partial_t v_z = \partial_z b_z + [b_z, \psi] + [\phi, v_z]. \quad (21)$$

In a low $\beta$ plasma, the role of the plasma compressibility is to make important contribution to the production of a $b_z$, the latter being produced solely by Hall effect in a large $\beta$ incompressible plasma. Following Fitzpatrick, we define $Z = b_z / c_B$, $c_\omega = \sqrt{\beta(1 + \beta)}$, and $d_p = c_B d_p$, then with these notations Eqs. (18)–(21) are rewritten as

$$\partial_t \psi = \partial_z (\phi - d_p Z) + [\phi - d_p Z, \psi], \quad (22)$$

$$\partial_t Z = \partial_z (c_\omega v_z - d_B b_z) + [c_\omega v_z - d_B b_z, \psi] + [\phi, Z], \quad (23)$$

$$\partial_t \omega_z = \partial_z j_z + [j_z, \psi] + [\phi, \omega_z], \quad (24)$$

$$\partial_t v_z = c_\omega \partial_z Z + [\phi, v_z] + c_\omega^2 Z, \quad (25)$$

This reduced model conserves the total energy

$$E = \int d^3 r (\nabla \phi)^2 + v_z^2 + (\nabla \psi)^2 + Z^2, \quad (26)$$

linearizing it leads to the following dispersion relation:

$$\omega^2 - k_z^2 (1 + c^2_\phi) + (k_\parallel d_p)^2 \omega^2 + k_z^2 = 0. \quad (27)$$

For $k_\perp d_p \ll 1$, the dispersion relations of the shear Alfvén waves and the slow waves, respectively, $\omega_{s z} = \pm k_z$ and $\omega_{s} = \pm c \omega_B$, are recovered. These two waves have the same phase velocity only when $\beta \gg 1$ ($c_B \sim 1$), the slow wave becoming a sound wave when $\beta \ll 1$. Taking the limit $\beta \rightarrow 0$ ($c_B \rightarrow 0$) while keeping $d_p$ finite in Eq. (12) gives the dispersion relation of the “kinetic” Alfvén waves, $\omega_{s z} = \pm k_z \sqrt{1 + \rho_i^2 k_z^2}$, with $\rho_i = \sqrt{\beta} d_i$ being the ion sound Larmor radius.

When $\beta \ll 1$ then $c_B = \sqrt{\beta}$, $d_p = \rho_i$, $Z = b_z / \sqrt{\beta}$, and the above set of equations is

$$\partial_t \psi = \partial_z (\phi - \rho_i Z) + [\phi - \rho_i Z, \psi], \quad (28)$$

$$\partial_t Z = \partial_z (\sqrt{\beta} v_z - \rho_i j_z) + [\sqrt{\beta} v_z - \rho_i j_z, \psi] + [\phi, Z], \quad (29)$$

$$\partial_t \omega_z = \partial_z j_z + [j_z, \psi] + [\phi, \omega_z], \quad (30)$$

$$\partial_t v_z = \sqrt{\beta} \partial_z Z + [\phi, v_z] + \sqrt{\beta} Z, \psi]. \quad (31)$$

This system is similar to the “four-field model” first derived by Hazeltine et al. These four-field models can be extended to account also for the effect of the electron inertia. It is now easy to consider how the slow mode, which turns into a sound wave with phase speed $C_s < V_A$ in a low $\beta$ plasma, is modified by two-fluid effects. In order for the kinetic Alfvén waves to remain at zero amplitude, we assume $\phi \sim \rho_i Z$, in which case $\partial_t \psi = 0$ from Eq. (28) and we can eliminate $j_z$ from the linearized Eqs. (29) and (30), $\partial_t (1 - \rho_i^2 \nabla_z^2) \phi = \partial_z (\sqrt{\beta} \rho_i \psi)$, which combined with Eq. (31) gives the following wave equation for the sound wave: $\partial_t (1 - \rho_i^2 \nabla_z^2) \phi = \partial_z \beta \phi$.

In the limit $\beta \rightarrow 0$, but $\rho_i$ finite, the parallel flow dynamics decouples, i.e., $v_z$ is passively advected and satisfies $\partial_t \psi = [\phi, v_z]$. In this limit, the previous four-field model reduces to

$$\partial_t \psi = \partial_z (\phi - \rho_i Z) + [\phi - \rho_i Z, \psi], \quad (32)$$

$$\partial_t Z = \rho_i \partial_z j_z - \rho_i [j_z, \psi] + [\phi, Z], \quad (33)$$

$$\partial_t \omega_z = [j_z, \psi] + [\phi, \omega_z]. \quad (34)$$

Here, $E_3 = \int d^3 r (\nabla \phi)^2 + (\nabla \psi)^2 + Z^2$ are separate conserved energies. In a uniform plasma this three-field model can be simplified further. Indeed, multiplying Eq. (33) by $-\rho_i$, we see from Eqs. (33) and (34) that $Z = -\rho_i \omega_z$, hence the following two-field model is obtained:

$$\partial_t \psi = \partial_z (\phi + \rho_i \omega_z) + [\phi + \rho_i \omega_z, \psi], \quad (35)$$

$$\partial_t \omega_z = [j_z, \psi] + [j_z, \psi]. \quad (36)$$

This model also conserves $E_4 = \int d^3 r (\nabla \phi)^2 + (\nabla \psi)^2 + \rho_i^2 \omega_z^2$. Notice that it is very similar to the electron-MHD system for $k \gg \rho_i$, or if we suppose that the ions play the role of a neutralizing background, while it reduces to the conventional RMHD description of shear Alfvén wave polarized perturbations for $k \ll \rho_i$. Linearizing this two-field model, Eq. (36) gives $\partial_t \phi = \partial_t \psi$, which combined with Eq. (35) yields the equation, $\partial_t \psi = \partial_z (1 - \rho_i^2 \nabla_z^2) \phi$, for the dispersive kinetic Alfvén waves with dispersion relation

$$\omega_{s z} = \pm k_z \sqrt{1 + \rho_i^2 k_z^2}. \quad (37)$$

Some physical properties of the kinetic Alfvén waves, which are believed to play a crucial role in the “dispersive range” of MHD turbulence, were studied in Ref. 8. The kinetic Alfvén wave is a compressive mode which, unlike the Alfvén wave, involves a parallel component of the magnetic field perturbation such that the total pressure perturbation vanishes. In this sense, the kinetic Alfvén wave also takes on some properties of the large $k_\perp$ limit of the slow mode. Also in Ref. 8, a two-fluid extension of the MHD dispersion relation was derived: $\left( \omega^2 / k_A v_A^2 - 1 \right) \left[ \omega^2 (\omega^2 - k_A v_A^2) - \beta k_A^2 v_A^2 (\omega^2 - k_A v_A^2) \right] \equiv \omega^2 (\omega^2 - k_A v_A^2)^2 \rho_i^2$, explicitly showing that for $\beta \ll 1$ the fast mode essentially cancels itself out of the dispersion relation.

We finally mention that the three-field model Eqs. (32)–(34) is the simplest fluid model that describes the non-
linear interactions between kinetic Alfvén waves and drift waves, if also the effect of a background plasma inhomogeneity, say $Z_0 = \alpha x$, is taken into account,

$$\partial_t \psi = \nabla_p (\phi - \rho_Z Z) - \rho_e \alpha_{Z} \partial_z \psi, \tag{38}$$

$$d_z Z = - \nabla_p j_z - \alpha_{Z} \partial_z \phi, \tag{39}$$

$$d_z \omega_z = \nabla j_z. \tag{40}$$

The notations $d A = \partial_t + [A, \phi]$ and $\nabla A = \partial A + [A, \psi]$ have been employed. As done above for the description of the sound waves, assuming $\phi - \rho Z$, we obtain from Eqs. (39) and (40) a familiar equation describing the nonlinear dynamics of electrostatic drift waves,

$$d_t (\phi - \rho Z \nabla^2 \phi) - v_s \partial_z \phi = 0, \tag{41}$$

with $v_s = \rho_e \alpha Z$.

$$\omega = - \frac{v_s k_x}{1 + \rho_e^2 k_x^2}, \tag{42}$$

being the dispersion relation of the drift wave.

In summary, we report the derivation of reduced models that describe the two-fluid dynamics of a plasma embedded in a strong uniform magnetic field. The starting point is the extension of the RHMHD system derived in Ref. 4, which accounts also for some aspects of the role of the plasma compressibility. Therefore, the domain of validity of this RHMHD system is not limited only to a large value of the plasma pressure parameter $\beta$. In a low-$\beta$ plasma, the role of the plasma compressibility is to make important contribution to the production of a perturbation in the guide field strength, contrary to the case of a large-$\beta$ incompressible plasma where the latter can solely be produced by Hall effect. Already at the level of standard RMHD, it is essential to retain this effect in order to account for how the slow wave becomes acoustic in a low-$\beta$ plasma. Moreover, compression is an essential physical property of the kinetic Alfvén wave, a two-fluid modification of the Alfvén mode, which involves a magnetic perturbation parallel to the background field and, hence, also takes on some properties of the large $k_\perp$ limit of the slow mode. The procedure, which has been followed here to obtain the RHMHD model, is identical to the one in Ref. 5 without the restriction to two-dimensional perturbations with $\partial_z = 0$. The resulting model being very similar in its theoretical structure to the incompressible RHMHD system of Ref. 4, this makes it a valuable computational tool for understanding topics such as the small “dispersive scales” of plasma turbulence and parallel electric field generation, for instance, in an astrophysical context. We discuss the point of contact which exists between this RHMHD system and other reduced Branginski models, which are used for investigating the dynamics of magnetically confined laboratory plasmas$^9$–$^{11}$ (the latter being stratified, their dynamics also involve drift waves). Among them, the “four-field model”$^6,^7$ plays a pivotal role and its Hamiltonian formulation, for the two-dimensional case, was discussed recently in Ref. $^{12}$.