THE STRUCTURE OF THE ACCRETION DISC BOUNDARY LAYER AROUND A ROTATING, NON-MAGNETIZED STAR

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Abstract. The process of disc accretion onto rotating nonmagnetized stars is considered. Closed system of equations is written for the boundary layer which exists between Keplerian outer part of the disc and more slowly rotating star surface. Under certain simplifying assumptions the system is solved numerically for main physical variables. Energy conservation equation is used for finding the boundary layer radiation intensity. The results are discussed with a special emphasize on their importance as well as on possible shortcomings.

1. Introduction

It is widely known that if accretion disc is formed around a stellar object with weak enough magnetic field, then its innermost region adjacent to the stellar surface cannot be Keplerian since the stellar angular velocity at its surface in the equatorial plane $\Omega_0$ must be less than the corresponding Keplerian value $\Omega_k(R) = (GM/R^3)^{1/2}$. The region is conventionally called Boundary Layer and hereafter it will be mentioned as BL. In the outer-Keplerian part of the accretion disc angular velocity increases with the decreasing of the distance from the star surface. Such an increase goes on up to the BL where accreted matter decelerates down to the $\Omega_0$ at the star surface which is inner boundary of the disc. Evidently, there always exists point where angular velocity attains its maximum value and begins to decrease. Although BL should be quite narrow, estimations show that up to about one half of the accretion luminosity may be generated in this region (Pringle, 1981). It means that consideration of BL has the principal observational importance for all those astrophysical objects where it exists.

The disc-star boundary layer and its influence on the accretion disc structure has been studied for almost as long as accretion discs in general. The simplest model for BL was considered by Lynden-Bell and Pringle (Lynden-Bell and Pringle, 1974) assuming constant turbulent viscosity and zero pressure in the layer. Pringle (1977) and Pringle and Savonije (1979) investigated BL structure in order to explain X-ray emission of dwarf novae and proposed the existence of strong shocks in BL. From the big enough subsequent number of publications of various authors concerning the subject we want to mention papers by Papaloizou and Stanley (1986) and by
Stanley (1988) where the authors give modified viscosity prescription for BL and assume polytropic equation of state between the surface density \( S \) and vertically integrated pressure \( W \) in the BL. They have found certain instabilities in the BL and pointed out at the possible relevance of them to QPO's in low mass X-ray binaries (Stanley, 1988).

It appears to be quite difficult to construct full theory of accretion disc boundary layer since it deals with the solution of the set of differential equations describing the conservation of the momentum and energy of the accreting matter in the presence of the developed turbulence. Moreover, the modeling of the turbulent viscosity being difficult enough in the Keplerian parts of accretion disc (Pringle, 1981), becomes even more difficult in BL due to the effects evoked by sharp gradients of various physical quantities in the vicinity of the stellar surface.

In the present paper we consider accretion disc boundary layer using modified form of the standard "\( \alpha \)-law" by Shakura and Sunyaev (1973) mentioned above. In order to exclude the thermodynamical treatment of BL we also adopt polytropic relationship between \( S \) and \( W \). Originally we derive the most general equations governing structure of BL: equations of mass, momentum and energy conservation and reduce them to the set of three ordinary differential equations. Using above mentioned simplifications, we further reduce the system to the solvable form and find the representative solutions for various plausible values of BL parameters. These solutions are compared with each other and the results are critically discussed. We have found solutions for the intensity of BL radiation by means of the equation of energy conservation—and compared obtained results with the existing qualitative notions.

### 2. Main Equations

In this paper we consider stationary and axisymmetric accretion disc, which is circulating around central star of mass \( M \) and equatorial radius \( R \). We use cylindrical coordinates \((r, \phi, z)\) with the \( z \)-axis chosen as the axis of rotation \((z = 0)\) is the equatorial plane of the disc). Main equations which govern the structure of BL are those of the conservation of mass, radial momentum, angular momentum and energy (Straumann, 1984):

\[
\dot{M} = -2\pi r S v_r = \text{constant}. \tag{2.1}
\]

\[
v_r \frac{\partial v_r}{\partial r} = r \left( \Omega^2 - \Omega_k^2 \right) - \frac{1}{S} \left( \frac{\partial W}{\partial r} \right), \tag{2.2}
\]

\[
S v_r \frac{\partial r^2 \Omega}{\partial r} = \frac{1}{r} \frac{\partial r^2 W}{\partial r}, \tag{2.3}
\]
\[ S v_r \frac{\partial}{\partial r} \left( \frac{v_r^2}{2} + \frac{r^2 \Omega^2}{2} + (A + 1) \frac{W}{S} + \Phi \right) = \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \Omega W_{\varphi r} \right) - Q^-; \]  

where all notations are standard. Namely, \( v_r \) is the radial component of mean velocity; \( \Omega \equiv v_{\varphi}/r \) is the angular velocity; \( W, S \) and \( W_{\varphi r} \) are vertically integrated pressure, surface density and turbulent viscosity tensor non-zero component respectively; \( \Omega_k \equiv \left[ G M/r^3 \right]^{1/2} \) is the Keplerian angular velocity; \( \Phi \equiv -r^2 \Omega_k^2 \) is the gravitational potential, \( Q^- \) is the energy flux per unit area emitted at the disc surface; \( \dot{M} \) is the mass accretion rate and \( A \) is a constant which will be defined below.

We shall write equation of state for the BL also in a standard way. In particular, the total pressure in the BL will be treated as the sum of the gas and radiation pressures

\[ P = P_g + P_{\text{rad}} = \frac{k}{\mu m_H} \rho T + \frac{a T^4}{3}. \]  

The internal specific energy is

\[ \varepsilon \equiv c_v T + \frac{a T^4}{\rho} \equiv \frac{A}{P/\rho} \]  

where

\[ A \equiv \frac{\beta}{\gamma - 1} + 3(1 - \beta); \]  

and \( \beta \equiv P/P_g \); \( \gamma \equiv c_p/c_v \). Note that when \( P \approx P_{\text{rad}} \) in the BL \( A = 3 \), while for the BL with \( P \approx P_g \), \( A = 3/2 \).

For the sole non-zero component of turbulent viscosity tensor, \( W_{\varphi r} \), we have the following expression (Straumann, 1984)

\[ W_{\varphi r} = r S \nu_t \frac{\partial \Omega}{\partial r}; \]  

where \( \nu_t \) is the kinematic coefficient of turbulent viscosity.

Let us define the sound speed \( c_s \) by the formula

\[ c_s^2 \equiv \frac{W}{S} \]  

when taking into account (2.1) we can rewrite (2.2–2.4) as

\[ v_r \left[ 1 - \frac{c_s^2}{v_r^2} \right] \frac{\partial v_r}{\partial r} = r(\Omega^2 - \Omega_k^2) - \frac{\partial c_s^2}{\partial r} + \frac{c_s^2}{r}, \]  

\[ v_r(r^2 \Omega - L_m) = r^2 \nu_t \frac{\partial \Omega}{\partial r}, \]  

\[ \]
\[ v_r \frac{\partial v_r}{\partial r} + r(\Omega_k^2 - \Omega^2) - (r^2 \Omega - L_m) \frac{\partial \Omega}{\partial r} + (A + 1) \frac{\partial c_s^2}{\partial r} = \frac{2\pi r}{M} Q^-; \tag{2.11} \]

where \( L_m \) is a constant of integration which will be defined below.

Note that equations (2.9–2.11) contain five unknown functions: the radial velocity \( v_r \), the angular velocity \( \Omega \), the sound speed \( C_s \), the viscosity coefficient \( \nu_t \), and the “cooling function” \( Q^- \). If we would have some model for the latter two quantities, expressing them by means of the other three ones

\[ \nu_t = F_1(v_r, \Omega, c_s), \]
\[ Q^- = F_2(v_r, \Omega, c_s); \tag{2.12} \]

then the system of equations would become closed and in principle, may be solved numerically with proper boundary conditions. But the very complex and difficult problem is specification of functions \( F_1 \) and \( F_2 \) since it is connected with the unknown processes of generation and transport of the radiation in the BL and with the even more hazardous enigma of turbulent viscosity in the strongly sheared accreting flow. Describing turbulence, being difficult enough in standard Keplerian accretion discs, becomes rather more difficult in essentially non-Keplerian BLs.

In the meanwhile radial drift velocity \( v_r \) in boundary layer is supposed to be subsonic. Otherwise the presence of the central stellar object cannot be communicated outwards to the accreting flow (Pringle, 1977). Another important assumption which we shall use in this paper is an expression for the kinematic viscosity coefficient \( \nu_t \):

\[ \nu_t = \frac{\alpha c_s^3}{c_s \Omega_k + \beta r(\Omega_k^2 - \Omega^2)}, \tag{2.13} \]

originally suggested in a slightly different form in Papaloizou and Stanley (1986). Here \( \alpha \) is a conventional constant coefficient of the accretion disc standard theory (0.01 \( \leq \alpha \) \( \leq 1 \)), and \( \beta \) is a constant of order unity. It is easy to notice that in the Keplerian part of the disc Eq. (2.13) reduces to the usual “\( \alpha \)-law”: \( \nu_t = \alpha c_s H \), (\( H \) is the half-thickness of the disc). However, in the BL the value of viscosity is noticeably less, corresponding to the shorter mean free path determined by the pressure scale-height in the radial direction (Papaloizou and Stanley, 1986).

In this paper, in order to avoid thermodynamical treatment of the disc, a polytropic relation between the vertically integrated pressure \( W \) and the surface density \( S \) is assumed, i.e.,

\[ W = k S^{1+1/n} \tag{2.14} \]

where \( k \) is a polytropic constant and \( n \) is the polytropic index (Papaloizou and Stanley, 1986). Adoption of Eq. (2.14) has one more important advantage: It allows us to resolve the system of equations without using of energy conservation
equation and after finding of main disc variables \((v_r, c_s, \Omega, W \text{ and } S)\) to find the radiation intensity \(Q^-\) through Eq. (2.11) without any assumptions about the radiation mechanism in the BL (i.e., without using of any model like that of \(Q^-\) in Eq. (2.12)).

First step in this direction is to write by means of Eqs. (2.14), (2.8) and (2.1) the following connection between the modulus of radial velocity and sound speed:

\[
|v_r| = \left( \frac{\dot{M} k^n}{2\pi} \right) \frac{1}{rc_s^{2n}}. \tag{2.15}
\]

The equation of radial momentum conservation Eq. (2.2) may be rewritten to get

\[
c_s \frac{\partial c_s}{\partial r} = -\frac{r(\Omega_k^2 - \Omega^2)}{2(n + 1)}. \tag{2.16}
\]

Substituting further Eqs. (2.15) and (2.13) into (2.10) we get

\[
\frac{\partial \Omega}{\partial r} = -\left( \frac{\dot{M} k^n}{2\pi} \right) \frac{(r^2 \Omega - r_m^2 \Omega_m) \left[ c_s \Omega_k + \beta r(\Omega_k^2 - \Omega^2) \right]}{\alpha r^3 c_s^{2n+3}}. \tag{2.17}
\]

where we have written \(L_m \equiv r_m^2 \Omega_m\). The physical meaning of such a notation is clear: As far as radial derivative of angular velocity \(\Omega(r)\) changes its sign at the point \(r_m\) where it attains its maximum value \(\Omega = \Omega_m\), we see that the constant of integration \(L_m\) in Eq. (2.2), which has the dimension of angular momentum, may be expressed as it is actually done in Eq. (2.17).

The system of differential equations (2.16–2.17) is already solvable since it is written for only two unknown functions \(\Omega(r)\) and \(c_s(r)\). Let us introduce the following dimensionless variables

\[
x \equiv \frac{r}{r_0} \\
\omega \equiv \frac{\Omega}{\Omega_k(r_0)} \\
c \equiv \frac{c_s}{v_{\varphi k}(r_0)} \\
v \equiv \frac{v_r}{v_{\varphi k}(r_0)}
\]

where \(v_{\varphi k}(r_0) \equiv r_0 \Omega_k(r_0)\).

Eq. (2.16) in these variables may be rewritten as

\[
\frac{\partial c}{\partial x} = \frac{x(w^2 - 1/x^3)}{2(n + 1)c}, \tag{2.19}
\]

and Eq. (2.17) may be reduced to
\[
\frac{\partial w}{\partial x} = -(v_0 c_0^2) \left( \frac{x^2 w - \varepsilon}{x^3} + \frac{\beta x (1/x^3 - w^2)}{\alpha x^3 c_0^{2n+3}} \right), \tag{2.20}
\]

where \( v_0 \equiv v(x = 1) \) and \( c_0 \equiv c(x = 1) \) are the values of the dimensionless radial velocity \( v(x) \) and the sound speed \( c(x) \) at the inner edge of the disc (star surface); and \( \varepsilon \equiv (r_m/r_0)^2 (\Omega_m/\Omega_k) \). Since the BL is thought to be quite narrow with the characteristic dimensionless width \( \Delta = (H/r_0)^2 \simeq c_0^2 \), it is clear that \( r_m \gtrsim r_0 \) and \( \Omega \lesssim \Omega_0 \) in such a way that \( \varepsilon = x_m^2 w_m^2 \simeq 1 \). Therefore, in the forthcoming consideration for simplicity we take \( \varepsilon = 1 \).

The equation of energy conservation (2.11) may be used now to express the cooling function (radiation intensity) \( Q^- \) by means of \( w \) and \( c \). After simple rearrangement of the terms we get

\[
q^- = \left( \frac{n - A}{n + 1} \right) \left( \frac{1}{x^3} - w^2 \right) - \frac{(x^2 w - \varepsilon)}{x} \frac{\partial w}{\partial x}, \tag{2.21}
\]

where we have introduced the dimensionless radiation function \( q^- \) defined as:

\[
q^- \equiv \frac{2\pi r_0^3}{G M \dot{M}} Q^- . \tag{2.22}
\]

The system of equations (2.19–2.20) may already be solved by means of some numerical method. For this purpose it remains to specify the values of parameters \( n, \alpha \) and \( \beta \) initial values of velocities \( v_0, c_0 \) and \( w_0 \). In the next chapter we discuss the method of solution and the solutions by themselves which were obtained for various values of these free parameters.

### 3. Method of Solution and Discussion of Results

The system of equations (2.19–2.20) contains several constant parameters which will be specified to solve the system numerically. One of the most important parameter is the angular velocity of the star. In dimensionless units stellar angular velocity \( \omega_0 \equiv \Omega(r_0)/\Omega_k(r_0) \) do not exceed the value \( \omega_0^\text{max} \approx 0.77–0.9 \) (Friedman et al., 1986), not even for rapidly rotating neutron stars. In the present consideration we perform numerical calculations for three values of stellar angular velocity, namely, \( \omega_0^{(1)} = 0, \omega_0^{(1)} = 0.2, \omega_0^{(1)} = 0.4 \).

It is evident that evaluation of initial values of the radial and sound velocities must be performed by means of Eqs. (2.19–2.20), written in the point \( r = R \) (that is \( x_0 = 1 \)). From Eq. (2.19) we see that the derivative of the sound velocity \( \partial c/\partial x \) is negative for arbitrary values of dimensionless distance \( x \), since \( w(x) < w_k(x) = 1/x^3/2 \) at any \( x-s \). Therefore, \( c(x) \) increases monotonically coming nearer to the stellar surface. Let us assume that at \( 1 \leq x \leq 1 + 2\Delta, \partial c/\partial x \approx -k e/x \). Inserting this expression into Eq. (2.19) we get

\[
c_0 \simeq \left[ \frac{1 - w_0^2}{2k(n + 1)} \right]^{1/2}. \tag{3.1}
\]
Fig. 1. Angular velocity $w(x)$ against distance from the star surface for different values of stellar angular velocity ($w_0^{(1)} = 0$, $w_0^{(2)} = 0.2$, and $w_0^{(3)} = 0.4$).

The upper line represents Keplerian angular velocity.

In the present treatment we choose $n = 1$ and $k = 100$ as representative values of these parameters. In this case $c_0^{(1)} = 0.05$, $c_0^{(2)} = 0.049$, $c_0^{(3)} = 0.0458$.

Now it remains to estimate the value of $v_0$ using Eq. (2.20). Originally we must estimate the value of the derivative $\partial w / \partial x$ at the inner edge of the BL (at point $x = 1$). The derivative may be calculated by the following rough expression:

$$\left( \frac{\partial w}{\partial x} \right)_{x=1} \approx \frac{w_m - w_0}{x_m - 1}. \quad (3.2)$$

As far as $x_m \approx 1 + (H/r)^2 \approx 1 + c_0^2$ and $w_m = L_m/x_m^2 \approx (1 + c_0^2)^{-2}$, Eq. (3.2) may be rewritten as

$$\left( \frac{\partial w}{\partial x} \right)_{x=1} \approx \frac{1 - w_0(1 + c_0^2)^2}{c_0^2(1 + c_0^2)^2}; \quad (3.3)$$

and inserting it into Eq. (2.20) written at the point $x = 1$ and expressing $v_0$ through other already known parameters we get

$$v_0 = \frac{\alpha c_0[1 - w_0(1 + c_0^2)^2]}{(1 + c_0^2)^2(1 - w_0)[c_0 + \beta(1 - w_0^2)]}. \quad (3.4)$$
Fig. 2. Sound velocity $c(x)$ against distance from star surface for several initial values ($c_0^{(1)} = 0.05$, $c_0^{(2)} = 0.049$, $c_0^{(3)} = 0.0458$).

for the values of $w_0$ and $c_0$ which we have chosen just above, we find that $v_0^{(1)} = 0.0047$, $v_0^{(2)} = 0.0048$, $v_0^{(3)} = 0.0051$.

The results of numerical calculations (we used a method due to Dormand and Prince with stepsize control) are represented by Figs. 1–3. In particular, in Fig. 1 the dependence of angular velocity $w(x)$ on dimensionless distance $x$ is shown for all three values of stellar angular velocity $w_0$. Here we have also drawn Keplerian angular velocity $w_k(x) = x^{-3/2}$ curve which looks like almost a straight line due to the narrowness of the BL width. After attaining their maximum values all three curves for angular velocities tend to the Keplerian curve as it should be. Such a behaviour may be treated as a strong argument in favour of our method of the boundary conditions determination.

In Figs. 2 and 3 we have represented the dependence of the sound velocity $c(x)$ and radial velocity $v(x)$ on the same variable $x$. It must be noted that the increasing of the sound velocity in the BL which is clearly seen in Fig. 2 for each value of initial sound speed $c_0$ leads to the increasing of the disc half-thickness in the BL since $H(x)/R \simeq c(x)x^{3/2}$. Such a widening of the BL to its bottom is in agreement with the qualitative expectations (Shakura and Sunyaev, 1988). radial velocity curves (see Fig. 3) are in resemblance with the behaviour of angular velocity. In particular, radial velocity increases as the BL is approached, reaches its maximum value nearby the point $x_m$ and decreases down to the value $v_0$. It is
Fig. 3. Radial velocity $v(x)$ against distance from the star surface for several initial values ($v_0^{(1)} = 0.0047, v_0^{(2)} = 0.0048, v_0^{(3)} = 0.0051$).

Fig. 4. Radiation intensity $q^-(x)$ against distance from the star surface for different values of stellar angular velocity ($w_0^{(1)} = 0, w_0^{(2)} = 0.2$ and $w_0^{(3)} = 0.4$).
worthwhile to note that Papaloizou and Stanley (1986) obtained similar result for the same (subsonic) kind of radial flow.

In Fig. 4 we have represented the behaviour of the radiation intensity function \( q^-(x) \) for the same three samples of boundary conditions. Calculation of \( q^-(x) \) is performed using Eq. (2.21) and the results of the solution of the system of differential equations (2.19–2.20). It is clearly seen that with the decreasing of the stellar angular velocity radiation intensity increases more rapidly and attains higher value at the inner edge of the disc. Thus it is evident that BL luminosity is maximum for the case of nonrotating star and decreases with the increase of \( u_0 \) as it also should be (Pringle, 1981).

We are fully aware that the rough theoretical model of the BL outlined in the present paper is of the qualitative value. Further refinement of the consideration with the usage of more accurate thermodinamical model and and assumptions about the character of the turbulence and radiative processes in BL needs separate consideration and is beond the scope of this paper.

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