Compressible hydromagnetic shear flows with anisotropic thermal pressure: Nonmodal study of waves and instabilities

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The evolution of linear magnetohydrodynamic waves and plasma instabilities (firehose and mirror) in a compressible, magnetized plane Couette flow with anisotropic thermal pressure is investigated. In the present study we revealed that the pressure anisotropy brings significant novelty to the effect of coupling and linear reciprocal transformation of the wave modes originally discovered [Chagelishvili, Rogava, and Tsiklauri, Phys. Rev. E 53, 6028 (1996)]. It is found that behavior of the firehose and mirror instabilities is drastically changed due to the presence of shear in the flow. These novel effects are caused by the non-normality of linear dynamics in shear flows and they have been revealed through use of the nonmodal approach. © 1997 American Institute of Physics. [S1070-664X(97)02303-3]

I. INTRODUCTION

The velocity field of almost all real astrophysical or laboratory hydrodynamic and hydromagnetic flows is commonly spatially inhomogeneous. When various adjacent layers of the flow move with different velocities (differential motion), the flow is said to be sheared. Feasible examples of astrophysical shear flows are differentially rotating fluid systems: planetary rings, stars and protostellar nebulae, accretion disks, and spiral galaxies. A complicated kind of three-dimensional shear motion is present in active galactic nuclei outflows (jets), stellar, and pulsar winds.

A variety of different types of waves and instabilities have been studied in astrophysical shear flows. Goldreich and Lynden-Bell2 invented the “shearing sheet model,” in order to discard purely geometrical complications, arising in flows with nontrivial geometrical symmetry, while retaining dynamical effects of the shear (differential rotation).3 In the framework of the model, the linear equations for perturbations can be rewritten in a comoving shearing reference frame, where they become temporarily inhomogeneous, just as in the nonmodal approach. In these coordinates, performing spatial Fourier expansion, the problem is reduced to the analysis of ordinary differential equations with time-dependent coefficients. In the framework of the “shearing sheet model,” the evolution of the galactic density waves has been investigated;4,5 the analytical study of shear instability in the Keplerian accretion disks has been undertaken;6 rapidly growing nonaxisymmetric, inviscid disturbances in the convectively unstable accretion disks have been found;7 and the interplay of Parker and shear instabilities in the accretion disks has been explored.8 Recent discovery of a powerful, local shear instability in weakly magnetized (high-β) accretion disks by Balbus and Hawley (originating from Refs. 9 and 10) created new interest in this field of study. Among their series of papers, the work (Ref. 11), where they applied shearing sheet formalism to the study of nonaxisymmetric modes in the disk is worthy of note. Besides, it has been shown12 that nonaxisymmetric (spiral) waves are subject to a similar, but physically different, instability occurring in low-β disks. In sum, the model has proved itself a good and convenient technique for solving a wide class of analogous problems, although its relation to the more usual and standard normal modes approach seems to be problematic.

As regards laboratory shear flows, it is worthwhile to note that the importance of nonmodal solutions of the magnetohydrodynamic (MHD) equations governing tokamak devices has also been revealed in previous contributions;13 in particular, the ballooning instabilities in tokamak devices with sheared toroidal flow. Based on the covering space concept,14 which, in fact, does not constrain any unstable solution to evolve as exp(iωt), it was shown that an adequate description of the ballooning mode can be performed considering the nonmodal solutions of the equations. Base-
cally, the ballooning-mode eikonal representation used by the author is similar to the “Kelvin formalism,” but its mathematical formulation is different, since it is applied to the physical system with axial symmetry.

However, it should be noted, that a change in the overall paradigm is currently taking place in the field of hydrodynamic stability (Ref. 15 and references therein). The traditional paradigm (normal modes approach) is eigenvalue analysis, which proceeds in two stages: (i) linearize about the regular (laminar) solution, and afterward (ii) look for unstable eigenvalues of the linearized problem. According to the standard theory, the unstable eigenvalue (i.e., the one from the complex upper half-plane) corresponds to the exponentially growing eigenmode of the linearized problem. The necessary condition for the flow to behave unstably is thought to be the existence of such eigenmodes. For some flows [e.g., thermally driven instabilities in the Rayleigh–Bénard convection flow, or centrifugally driven instabilities in rotating Couette (Taylor–Couette) flow] this conclusion is in good agreement with experimental results. At the same time, for the other kind of hydrodynamic flows, especially those driven predominantly by shear forces, the predictions of the normal modes approach fail to match most experiments. For Couette flow, for instance, instabilities are observed to “switch on” for Reynolds numbers as low as $Re=350$, while the common eigenvalue analysis predicts stability for all $Re$-s. Traditionally, this anomaly was recognized as a failure of linearization and was attributed to non-linear effects. It has been recently argued, however, that the failure of the normal modes analysis should be attributed to step (ii). As is stated in Ref. 15, “It is a fact of linear algebra that even if all of the eigenvalues of a linear system are distinct and lie well inside the lower half-plane, inputs to that system may be amplified by arbitrarily large factors if the eigenfunctions are not orthogonal to one other.” According to the definition, a matrix, or an operator, whose eigenfunctions are orthogonal is said to be “normal,” and while the operators arising in the Rayleigh–Bénard and Taylor–Couette problems fall in this category, the operators that arise in Poiseuille and Couette flow are not normal. That is why small perturbations to these flows may be amplified by factors of thousands, even when all the eigenvalues are in the lower half-plane. Hence, the use of the full spectral (Fourier or Laplace) expansion in such a category of problems may be misleading in this context.

Turning now, again, to the nonmodal approach, it should be noted that its application to the study of the hydrodynamic instabilities in shear flows, which originated with Lord Kelvin, has recently become well established and extensively used. In the framework of this formalism, one considers temporal evolution of spatial Fourier harmonics (“Kelvin modes”) of the perturbations without any spectral expansion in time. The wavenumber of each spatial Fourier harmonic (SFH) along the mean flow shear varies in time, in the linear approximation there exists a “drift” of the SFH in the plane of wave numbers ($k$ space). The method establishes itself as an effective and convenient tool in the study of the wide range of physical processes taking place in shear hydrodynamic and hydromagnetic flows, including anomalous processes of energy exchange between the mean flow and the perturbations—the shear energy drawing by SFH in these flows.

Until now, the nonmodal approach was predominantly applied to the study of incompressible and compressible hydrodynamic shear flows, or incompressible MHD shear flows. In particular, the authors of Ref. 22 demonstrated the “transient growth” of SFH in the incompressible two-dimensional (2-D) Couette flow. In Ref. 23, consideration of the magnetized (MHD) analog of the same flow revealed the existence of anomalous amplification of the SFH, corresponding to the slow magnetosonic mode. In Ref. 24, the evolution of 2-D SFH in a compressible, plane Couette flow was studied. A new mechanism of the energy exchange between the mean flow and sound-type perturbations was found, leading to the nonexponential, monotonous growth of the SFH energy.

Recently, in the framework of the nonmodal approach, and based on the results from Ref. 27, the authors of Ref. 25 investigated the processes of energy exchange of Alfvén, slow, and fast magnetosonic waves with three-dimensional, magnetized, isotropic Couette flow, and also considered mutual transformations of these waves. The same kind of processes, but in electron–positron MHD plasma, were studied in Ref. 26.

In the present study, we apply the nonmodal approach to the consideration of compressible, magnetized free shear flow with anisotropic thermal pressure. For the sake of generality, we shall adopt the most general equations of state for a MHD medium. Some important and preliminary results of this study concerning the effect of coupling and linear transformation of hydromagnetic waves in shear flows, are represented, in part, in Ref. 27. In the present paper, after deriving the general set of linearized equations for perturbations, we shall investigate the case of strongly magnetized nonrelativistic collisionless plasma flow. The pressure of the medium will be taken as anisotropic. The physical influence of the shear on the MHD waves, as well as its interaction with plasma instabilities (firehose and mirror instabilities) in the flow, will be considered.

Before we start the presentation of our results, we would like to mention that in this paper, in contrast to Ref. 27, we consider a flow with an inhomogeneous regular velocity profile (linear shear) in a collisionless plasma described by the double adiabatic theory of Chew, Goldberger, and Low. At first glance it is more usual to expect that consideration of a flow with regular velocity inhomogeneity is conceivable exclusively in the collisional medium, where the collisions of constituent particles ensure momentum transfer and in turn, onset of the shear. However, it must be pointed out that the investigation of the physical processes in collisionless plasma with an inhomogeneous velocity profile (shear flow) originated in the mid-1960s. Furthermore, thermo-nuclear applications of the drift Kelvin–Helmholtz instabilities in the same kind of plasma flows were studied in Refs. 36 and 37, and the space-plasma applications were considered in Refs. 38–40. For even more detailed references on this issue, see, e.g., Ref. 41, or for a general discussion, see the textbook by Mikhailovskii. Mikhailovskii argued that
the use of the MHD equations for the medium under discussion requires care and prudence, since in the case of a collisionless plasma the equations do not give the full pattern of the instabilities, because they do not allow for the essentially kinetic effects (clearly, the hydrodynamic approach has a narrower region of validity than the kinetic one). In any event, consideration of the physical processes in the collisionless plasma with an inhomogeneous velocity profile is very well established in the field. One can argue, however, that in the absence of collisions in the medium there is no means to transfer the momentum across the flow; even taking into account finite Larmor radius effects \(^{44}\) does not resolve the problem. Putting this in other words, in the absence of collisions, it is impossible to accomplish onset of a velocity shear via the momentum transfer across the flow. Whereas in collisions, it is impossible to accomplish the onset of a velocity shear via the momentum transfer across the flow. However, if the unperturbed velocity inhomogeneity in the flow is taken for granted, as, for instance, in solar and/or pulsar winds, then our model seems to be fully relevant. Obviously, when the particles are ejected from the highly convective solar corona, further traveling along the open magnetic field lines, nothing prevents them from having different injection velocities, which will be maintained thereafter due to the absence of collisions. Another essential argument in favor of our model is the presence of a strong regular magnetic field in the flow along with its “frozen in” condition [Eq. (5)]. In our description of the shear, the particles can move all the way only along a particular field line. Therefore, if the particles moving along the two adjacent field lines initially had different velocities (due to any external means, e.g., nonuniform \(\nabla P_{0,1}\) — unperturbed parallel pressure gradient—which causes the flow itself at the origin of the flow), this difference will be maintained because of the absence of collisions. The latter unambiguously justifies the relevance of the consideration of the shear in a collisionless flow with a magnetic field.

Finally, we would like to emphasize that in our actual calculations we restrict ourselves to the consideration of only linear regular velocity shear in the unbounded flow. However, it is clear that every flow with a nonlinear (smooth) shear profile can be effectively represented by superposition of thin layers with linear shear.\(^\text{29}\) This ensures that our model could be relevant for various types of realistic astrophysical or laboratory flows.

Section II contains the derivation of the main equations governing the physical system under consideration in the framework of the nonmodal approach. In Sec. III we focus our attention on the 2-D case. In subsections we separately demonstrate cases of single and double transformations, and study novel effects due to the firehose and mirror instabilities. In the concluding section of the paper, we discuss the physical importance of the obtained results and their applicability to the above-mentioned kinds of realistic shear flows.

II. MAIN CONSIDERATION

The basic system of equations, governing the physics of the magnetized plasma flow with anisotropic thermal pressure, can be written in the following way:

\[
\partial_t \rho + \text{div}(\rho \mathbf{u}) = 0, \tag{1}
\]

\[
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \text{grad} \left( P_{\perp} + \frac{B^2}{8\pi} \right) - \frac{1}{\rho} \left( \mathbf{B} \cdot \nabla \right) \left[ \left( -\frac{1}{4\pi} + \frac{P_\parallel}{B^2} \right) \mathbf{B} \right], \tag{2}
\]

\[
\text{div} \mathbf{B} = 0, \tag{3}
\]

\[
\partial_t \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{B} - \mathbf{B} \cdot \text{div} \mathbf{u}. \tag{4}
\]

The latter equation—magnetic field induction equation—may also be written in the form of the so-called \textit{Walen equation},\(^{35}\) which will be useful below,

\[
\left[ \partial_t + (\mathbf{u} \cdot \nabla) \right] \frac{\mathbf{B}}{\rho} = \left( \frac{\mathbf{B}}{\rho} \cdot \nabla \right) \mathbf{u}. \tag{5}
\]

Here all notations are standard, namely, \(\mathbf{u}\) stands for the plasma velocity, \(\mathbf{B}\) is the magnetic field, and \(\rho\) represents matter density.

The equations of state in the most general form may be represented by the following pair of expressions:\(^{30,31}\)

\[
P_{\perp} = c_{\perp} \rho B^2, \tag{6a}
\]

\[
P_{\parallel} = c_{\parallel} \left( \frac{\rho}{B} \right)^{2a}, \tag{6b}
\]

where \(P_{\perp}\) and \(P_{\parallel}\) stand for the perpendicular and parallel thermal pressures in respect to the external magnetic field. Note that in (6), \(c_{\parallel}\) and \(c_{\perp}\) are some dimensional constants. When \(P_{\parallel} = P_{\perp}\) and polytropic indices \(a = \delta = 0\), the equations of state reduce to conventional equations of state for an isothermal ideal gas \(P \sim \rho\). For the anisotropic pressure case, when \(P_{\perp} \neq P_{\parallel}\) and \(a = \delta = 1\) (which, actually, will be used in all our numerical work), they represent the usual Chew–Goldberger–Low equations of state,\(^{32}\) while when \(\alpha = \delta = -\frac{1}{2}\), we obtain the equations of state for a collisionless, strongly magnetized plasma with ultrarelativistic temperature.\(^{31}\)

Let us consider the three-dimensional (3-D) compressible shearing sheet flow with the regular magnetic field \(\mathbf{B}_{||}|| \mathbf{u}_{0}\) and mean velocity \(\mathbf{U}_{0} = (U_{0x}, A_{0}, 0, 0)\) (the \(X\) axis is directed along the regular velocity vector, the \(Y\) axis is directed along a gradient of the shear), where the constant \(A\) (without loss of generality, we adopt \(A > 0\)) parametrizes the regular velocity shear.

To perform linearization of the system (1)–(5), we set \(\mathbf{u} = \mathbf{U}_{0} + \mathbf{u}'\), where \(\mathbf{u}'\) denotes perturbation of the velocity. The basic system of linearized equations describing the evolution of the small-scale, 3-D perturbations in this flow then is as follows:

\[
(\partial_t + A_y \partial_x) d + \partial_x u_y^1 + \partial_x u_z^1 = 0, \tag{7}
\]

\[
(\partial_t + A_y \partial_x) u_x^1 + A u_y^1 = -C_{\perp}^2 \partial_x S_i + (C_{||}^2 - C_{\perp}^2) \partial_x b_x, \tag{8}
\]
\[
(\partial_t + Ay \partial_x) u_y^1 = -C_2^1 \partial_y S_2 + C_2^2 \partial_y b_y - \partial_y b_x, \\
(\partial_t + Ay \partial_x) u_z^1 = -C_2^1 \partial_z S_2 + C_2^2 \partial_z b_z - \partial_z b_y, \\
(\partial_t + Ay \partial_x) b_y = \partial_y u_1^1, \\
(\partial_t + Ay \partial_x) b_z = \partial_z u_1^1, \\
\partial_t b_x + \partial_y b_y + \partial_z b_z = 0, 
\]
where \(d = \rho_0/\rho_0 \) (\( \rho_0 \) const), \( b = B'/B_0 \) (\( B_0 \) const), \( C_2^2 = P_0/\rho_0, C_2^1 = P_0/\rho_0 \), and \( C_\lambda \) is an Alfvén velocity (here and afterward physical quantities with primes denote corresponding perturbations, whereas the ones with subscripts or superscript 0 represent the unperturbed quantities). The quantities \( S_2 = P_0'/P_0 \) and \( S_1 = P_0'/P_0 \) may be expressed through perturbations of density and magnetic field with the help of the equations of state:

\[
S_1 = d + \partial b x, \\
S_i = (2\alpha + 1)d - 2\alpha b_x. 
\]

The nonmodal analysis implies the following substitution of variables:22–27 \( x_1 = x - Ay t; y_1 = y; z_1 = z; t_1 = t. \) Doing so we can rewrite Eqs. (7)–(13) in the following form:

\[
\partial_{t_1} u^1 + \partial t_1 u^1 + (\partial_y - A t_1 \partial_{x_1}) u^1 + \partial_{z_1} u^1 = 0, \\
\partial_{t_1} u_1^1 + A u_1^1 = -C_2^1 \partial_{z_1} S_2 + (C_2^2 - C_2^1) \partial_{x_1} b_x, \\
\partial_{t_1} u_2^1 = -C_2^2 (\partial_y - A t_1 \partial_{x_1}) S_2 + C_2^1 (\partial_y - A t_1 \partial_{x_1}) b_x + (C_2^3 - C_2^2) \partial_{y_1} b_y, \\
\partial_{t_1} u_3^1 = -C_2^2 \partial_{y_1} S_2 + C_2^3 \partial_{z_1} b_z + (C_2^3 - C_2^1) \partial_{z_1} b_z, \\
\partial_{t_1} b_y = \partial_{y_1} u_1^1, \\
\partial_{t_1} b_z = \partial_{z_1} u_1^1, \\
\partial_{t_1} b_x + (\partial_y - A t_1 \partial_{x_1}) b_x + \partial_{z_1} b_z = 0. 
\]

Let us perform the Fourier analysis of Eqs. (15)–(20), expanding the unknown functions denoted below, altogether, by \( F \), with respect to only the spatial variables, \( x_1, y_1, \) and \( z_1 \).

\[
F = \int dk_x dk_y dk_z \hat{F}(k_x, k_y, k_z, t) \exp[i(k_x x_1 + k_y y_1 + k_z z_1)], 
\]
where under \( F \) we imply all unknown functions appearing in (15)–(20). Further, using Eq. (14) and the Fourier transform of Eq. (20), we can obtain

\[
D^{(1)} = u_x + \beta(\tau) u_y + \gamma u_z, \\
u_x^{(1)} = -R u_y - (2\alpha + 1) D - [(2\alpha + 1) - \epsilon^2] \times [\beta(\tau) b_y + \gamma b_z], 
\]

where hereafter \( F^{(n)} \) will denote the \( n \)th-order time derivative of \( F \) and \( D = i\dot{u}, b = i\dot{b}_x, b_\gamma = i\dot{b}_z, R = A/lC_\lambda k_z, \sigma^2 = (C_\lambda/C_\beta)^2, \gamma^2 = (C_\lambda/C_\beta)^2, \), \( \beta(\tau) = k_1/t_1 - \sigma^2, \epsilon = C_\lambda/t_1, \gamma = k_1/t_1, v_\gamma = \dot{u}_1/lC_\lambda(i=x, y, z). \)

Note that the wave number of a SFH along the flow shear \( k_1(\tau) = k_1 - R k_1 \tau \) varies in time. This process of the linear drift of SFH in \( k \) space below will be referred as “the linear drift.” It brings a qualitative novelty in the evolution of perturbations in the shear flow (discussed in detail in Refs. 22 and 23).

If we introduce a new variable, \( \psi = D + \beta(\tau) b + \gamma b_\gamma \), we can reduce the system (22)–(27) to three intercoupled ordinary differential equations of the second order:

\[
\psi^{(2)} + \omega_1^2 \psi = \chi_1(\tau) b + \chi_2 b_\gamma, \\
\psi^{(2)} + \omega_2^2 b = \chi_1(\tau) \psi + \chi_3(\tau) b_\gamma, \\
\psi^{(2)} + \omega_3^2 b_\gamma = \chi_2 \psi + \chi_3(\tau) b. 
\]

We then introduce the following auxiliary notations:

\[
\omega_1 = \sqrt{2\alpha + 1}, \\
\omega_2(\tau) = \sqrt{(\epsilon^2 + \sigma^2 - 1) + [(1 + \delta) \epsilon^2 + \sigma^2] \beta(\tau)^2}, \\
\omega_3(\tau) = \sqrt{(\epsilon^2 + \sigma^2 - 1) + [(1 + \delta) \epsilon^2 + \sigma^2] \gamma^2}, \\
\chi_1(\tau) = \epsilon^2 \beta(\tau), \\
\chi_2 = \epsilon^2 \gamma, \\
\chi_3(\tau) = -[(1 + \delta) \epsilon^2 + \sigma^2] \gamma \beta(\tau). 
\]

Note that the quantity \( \psi \) is, actually, the \( x \)th component of a perturbation of the “Walen vector” 45 \( B/\rho_0 \), and the first-order differential equation, \( \psi^{(1)} = -R b + u_x \), is directly derived through the \( x \)th component of the Walen equation (5).

Further consideration becomes much more standard, since equations of this type are well known in the general theory of oscillations.46,47 They describe coupled oscillations with three degrees of freedom. The uncoupled eigenfrequencies and coupling coefficients appearing in (31)–(36) are \( \omega_i \) and \( \chi_i (i=1, 2, 3) \), respectively. The presence of shear in the flow \( (R \neq 0) \) ensures temporal variability of some of these quantities. However, their dependence on time may be considered as adiabatic when \( R \ll 1.24 \) Generally speaking, the theory of coupled oscillations is applicable to quite a wide class of different physical systems; Examples of these are mechanical systems with \( n \) > 1 degrees of freedom consisting of coupled oscillators; various kinds of molecules consisting of several atoms; oscillatory circuits with capacitive or inductive coupling; etc.46,47 It is widely acknowledged that the
existence of the coupling leads under certain circumstances to energy exchange between the oscillators and the transformation of fundamental oscillations into each other.

Hydrodynamic and other types of plasma flows constitute one such kind of oscillatory system, where coupling, as a rule, is associated with nonlinear processes (wave decay processes,50), which ensures the mutual transformation of different wave modes. In plasma physics, a linear coupling phenomenon is also known: mutual transformation of different kinds of plasma waves arising due to a spatial inhomogeneity of a medium. For example, the existence of the density inhomogeneity induces coupling between MHD oscillations,39,50 or, the transformation of compressional-type density discontinuity,51 etc. However, we must emphasize that the nature of the wave transformation effect discussed in this paper (which originally was discovered in Ref. 27) qualitatively differs from the already known linear transformation mechanisms.39–51 Density inhomogeneity-induced mode transformation occurs permanently in the limited spatial area (across the density inhomogeneity), while in our case transformation of the linear waves occurs in the whole volume filled by the flow, in the limited time interval (see below).

III. 2-D “SHEARING SHEET”

Hereafter we shall consider the plane (X0Y) “shearing sheet.” Evidently, the system (22)–(27) reduces, in this case, to the following set of four first-order differential equations:

\[ D^{(1)} = v_x + \beta(\tau)v_y, \]

\[ v_x^{(1)} = -Rv_y - (2\alpha + 1)D - [(2\alpha + 1) - \varepsilon^2] \beta(\tau)b, \]

\[ v_y^{(1)} = -\varepsilon^2 \beta(\tau)D + [(\varepsilon^2 + \alpha^2 - 1) + (\delta \varepsilon^2 + \alpha^2) \beta^2(\tau)]b, \]

\[ b^{(1)} = -v_y, \]

while the corresponding set of coupled second-order ordinary differential equations takes the following form:

\[ \psi^{(2)} + \omega_1^2 \psi = \chi_1(\tau)b, \]

\[ b^{(2)} + \omega_2^2 b = \chi_1(\tau)\psi, \]

where \( \omega_1 \) and \( \omega_2 \) are the same as in (31) and (32), but with \( \gamma = 0 \). Fundamental frequencies of corresponding coupled (normal) oscillations are \( \omega_1, \omega_2 \), and \( \Omega_1, \Omega_2 \) correspond to slow and fast magnetosonic waves (SMW and FMW), respectively. Since the oscillatory system described by (41)–(42) has two degrees of freedom, its behavior may be determined by two functions, \( \psi(\tau) \) and \( b(\tau) \). Note that all physical quantities from (37)–(40) may be expressed through \( \psi, b \), and their first derivatives.

As regards the total energy density of a particular spatial Fourier harmonic (SFH), we start its derivation from the standard expression for energy density in the physical space:

\[ e_{\text{tot}} = \rho_0 \left( \frac{u_1^2}{2} + \frac{u_2^2}{2} \right) + P_{\perp} + P_{\parallel} + \frac{B^2}{8\pi}, \]

where the first term on the right-hand side stands for the kinetic part of the total energy and the second and the third ones for the internal (compressional) and magnetic contributions to the total energy, respectively.

Further, using equations of state and retaining only perturbational terms of the second order, we can obtain the following expression for the total energy of the perturbations in the physical space:

\[ e_{\text{tot}} = \rho_0 \left( \frac{u_1^2}{2} + \frac{u_2^2}{2} \right) + P_{\perp} + \frac{3}{2} \left( \frac{\rho^2}{\rho_0} \right) b_x + \frac{3}{2} \left( \frac{\rho^2}{\rho_0} \right) b_y + \frac{3}{2} \left( \frac{\rho^2}{\rho_0} \right) b_z \]

\[ + \frac{3}{2} \left( \frac{\rho^2}{\rho_0} \right) b_z \]

\[ + \left( \frac{3}{2} \left( \frac{\rho^2}{\rho_0} \right) b_x + \frac{3}{2} \left( \frac{\rho^2}{\rho_0} \right) b_y \right) \left( \frac{b_x + b_y}{2} \right) \]

\[ + \left( \frac{3}{2} \left( \frac{\rho^2}{\rho_0} \right) b_y + \frac{3}{2} \left( \frac{\rho^2}{\rho_0} \right) b_x \right) \left( \frac{b_y + b_x}{2} \right) \]

Finally, normalizing \( e_{\text{tot}} \) by \( \rho_0 C_1^2 \), writing it for Fourier transforms of physical quantities, and rearranging the terms, we arrive at energy density of a particular SFH in the \( k \) space:

\[ E_{\text{tot}}(\tau) = \left[ \frac{v_1^2 + |v_2|^2}{2} + \frac{3}{2} |D|^2 - (\varepsilon^2 - 3) \beta(\tau)Db \right] + (\varepsilon^2 + \sigma^2 - 1) \frac{|b|^2}{2} + (3 + \sigma^2) \frac{\beta^2(\tau)|b|^2}{2}. \]

Here, we must emphasize the following. Equation (46) represents the total energy density of a particular SFH, in other words, the single harmonic of the Fourier expansion (21). However, a realistic wave packet may be composed of a number (even an infinite amount) of SFHs. Below, for brevity, we shall use the term “wave” instead of the “SFH of a wave,” but while simultaneously keeping this remark in mind.

Note that, strictly speaking, \( E_{\text{tot}}(\tau) \) is not a positively defined physical quantity. For instance, in the case when the condition for firehose instability \( (\varepsilon^2 + \sigma^2 < 1) \) is satisfied, the second term from the end of Eq. (46) is negative. The mirror instability also requires large values of \( \varepsilon^2 \) (see below), which presumably can make the third term negative. We will turn back to this point when we will discuss our numerical results concerning these instabilities. Besides the mathematically strict derivation of \( E_{\text{tot}}(\tau) \), our confidence in the correctness of its form is reinforced by the fact, that in all our numerical results (see below), the \( E_{\text{tot}}(\tau) \) curve is fairly smooth, although all contributing physical quantities are rather oscillatory (see, for instance, Fig. 21 and the corresponding energy curve in Fig. 22).

A. Stable oscillatory modes

In this section we consider only stable oscillatory modes excited in the flow under study. We demonstrate here single and double transformations of magnetosonic waves due to the linear mechanism found by us. First, let us focus our attention on those conditions that must be satisfied in order
to make the mutual transformation of the fundamental oscillatory modes, $\Omega_1(\tau)$ and $\Omega_2(\tau)$, possible. In fact, these conditions, but for *isotropic*, compressible magnetized shear flows, were first outlined in Ref. 27. Basically, the transformation conditions remain the same for the *anisotropic* shear flow case as well, but the latter case is much more valuable, since it involves new phenomena such as firehose and mirror instabilities, which may be of importance, for instance, in laboratory fusion devices or MHD generators or radiation mechanisms from various cosmic objects, etc. For completeness, we present here the conditions,27,47 the fulfillment of which ensures the mutual transformation of SMWs and FMWs:

(a) A so-called ‘‘degeneration region’’ (DR) should exist, where $|\omega_2^2 - \omega_1^2| = |\chi_1(\tau)|$, and

(b) The DR should be ‘‘passed’’ slowly—in a time interval sufficiently exceeding the beating period corresponding to $\chi_1(\tau);\omega_2(\tau) = |\chi_1(\tau)|$.

The exact mechanical analogy of the oscillatory system, governed by the same kind of equations, is flowing. Let us consider two coupled pendulums: the first one with constant uncoupled eigenfrequency $\omega_1 = \sqrt{2}\alpha + 1 = \sqrt{3}$, and the second one whose uncoupled eigenfrequency $\omega_2(\tau)$ is slowly (adiabatically) varied by some external means (e.g., a variable length). The interpendulum coupling coefficient $\chi_1(\tau)$ is also time dependent. As regards the condition (a), it is important to note the following: Bearing in mind that the mechanical analogy will partially lose its clarity when it comes to the consideration of unstable oscillatory modes (firehose and mirror instabilities), it is worthwhile outlining a graphical representation of the condition (a), while considering the dispersion curves $\Omega_1(\tau)$ and $\Omega_2(\tau)$. [Here we use the term dispersion curve since $\tau$ is related to $\beta(\tau)$, which itself is related to the angle $\theta$ between $\mathbf{B}_0$ and the wave vector $\mathbf{k}$ by the simple relation $\beta(\tau) = \tan \theta$.] Again invoking the mechanical analogy, it is reasonable to state that, in the case of weak coupling, the maximum energy exchange (i.e., transformation) between the pendulums with fundamental eigenfrequencies $\Omega_1(\tau)$ and $\Omega_2(\tau)$ occurs when they have the same length. In other words, when $\omega_2(\tau) = \omega_1$ and if the coupling between the pendulums is weak, then this implies that $\Omega_2(\tau) = \Omega_1(\tau)$ [or, i.e., the root in Eq. (43) becomes negligible]. Finally, we conclude that in the case of weak coupling, the transformation of the oscillatory modes occurs when their corresponding fundamental eigenfrequencies $\Omega_1(\tau)$ and $\Omega_2(\tau)$ approach each other, therefore forming the DR. In addition, the condition (b) also must be satisfied. However, as we will see below in the example of mirror instability, the latter graphical criterion, or it is better to say hint, is not generally true in the case of strong coupling.

1. Case of single transformation

Since we are studying a two-dimensional, compressible, and magnetized flow, it is clear that in such a medium there can exist two oscillatory modes (fast and slow magnetosonic waves, FMW and SMW), whose dispersion relation is given by (43). Based on the method given in Ref. 46, it is possible to set such initial conditions for Eqs. (37)–(40), so as to excite a distinct wave mode. Further, due to the presence of shear, as we mentioned above, this distinct mode [namely $\Omega_1(\tau)$ or $\Omega_2(\tau)$] will evolve in time. And, if the transformation conditions are satisfied, these two wave modes can undergo reciprocal transformations. For example, in Figs. 1–5 we present numerical solutions of Eqs. (37)–(40). We set initial conditions in such a way that initially SMW [i.e., $\Omega_1(\tau)$] was excited. All these graphs unambiguously show that the slow magnetosonic wave (SMW) has been trans-

![Figure 1](image1.png)

**FIG. 1.** Numerical solutions of Eqs. (37)–(40) for the set of the following physical parameters: $\alpha = \delta = 1$, $\beta(0) = 5$, $R = 0.05$, $\sigma = 1$, and $\epsilon = 3$. The solution was obtained with precise initial conditions $D(0) = 0.057 453 183$, $u(0) = 0.100 425 47$ $v(0) = -0.008 509 363 5$, and $b(0) = 0.008 509 363 5$ so as to distinctly excite the $\Omega_1(\tau)$ SMW mode. This set of figures (Figs. 1–5) represents a single transformation of SMW into FMW (see the details in the text). Such a high precision in the initial conditions is necessary to achieve exact separation of the wave modes, or in other words, to excite exactly a particular wave mode.

![Figure 2](image2.png)

**FIG. 2.** Numerical solutions of Eqs. (37)–(40) for the set of the following physical parameters: $\alpha = \delta = 1$, $\beta(0) = 5$, $R = 0.05$, $\sigma = 1$, and $\epsilon = 3$. The solution was obtained with precise initial conditions $D(0) = 0.057 453 183$, $u(0) = 0.100 425 47$ $v(0) = -0.008 509 363 5$, and $b(0) = 0.008 509 363 5$ so as to distinctly excite the $\Omega_1(\tau)$ SMW mode. This set of figures (Figs. 1–5) represents a single transformation of SMW into FMW (see the details in the text). Such a high precision in the initial conditions is necessary to achieve exact separation of the wave modes, or in other words, to excite exactly a particular wave mode.
formed into the fast magnetosonic wave (FMW). The effect of the transformation of SMW into FMW is rather pronounced in Figs. 1, 3, and 4, where we see that, initially, oscillations have low frequency and their amplitude is almost constant, as it should be, since $\Omega_1(\tau)$ far from the degeneration region (DR) is significantly smaller than $\Omega_2(\tau)$, and remains almost constant (see Fig. 6). However, after passing the DR the frequency of the oscillation increases drastically and the amplitude starts to increase. The transformation can be clearly seen in Fig. 6, where we plotted the following quantities: $E_{tot}(\tau)/E_{tot}(0)$, $\Omega_1(\tau)/\Omega_1(0)$, and $\{\Omega_2(\tau)/\Omega_2(100)\} \times [\Omega_1(100)/\Omega_1(0)]$. The latter quantity is rescaled so as to reveal the true dispersion of the SMW and FMW. It can be gathered from the graph that initially $E_{tot}(\tau) \sim \Omega_1(\tau)$, but after passing the point $\tau = \tau_{DR} = R/100$ (the midpoint of DR), where $\Omega_1(\tau)$ and $\Omega_2(\tau)$ approach each other, the SMW is transformed into the FMW. But there still remains a small admixture of SMW, which explains the inexact proportionality of $E_{tot}(\tau)$ with $\Omega_2(\tau)$. Such behavior of energy when the
waves evolve so that $E_{\text{tot}}(\tau) \sim \Omega_{1,2}(\tau)$ can be easily explained by the fact that all physical quantities of the oscillatory system vary in time adiabatically because $R \ll 1$. Since the parameters of the oscillatory system (adiabatically) vary in time, this means that the system is not a closed physical system. In turn, after the transformation, the energy of FMW is quasi-linearly increasing with time, which implies that the growth of the wave occurs at the expense of the shear energy of the mean flow. We point out that we purposefully choose the demonstration of the transformation of the SMW into the FMW. Indeed, the reverse process could take place if we initially excited the FMW, which certainly would transform into SMW as both conditions (a) and (b) are well satisfied for the set of parameters considered in this example. But, it is important to mention that the energy of the FMW is quasi-linearly increasing in time, so, certainly at some stage, nonlinear effects will switch on, triggering turbulence in the flow. Therefore, this new transformation mechanism could be helpful for understanding the stability laboratory flows, although one should bear in mind that this is a rather idealized picture. In a realistic physical situation, for instance in tokamaks, the flows have a much more complicated geometry, and velocity shear cannot be simply described in the framework of mere model of plane Couette flow. However, on a small scale, every flow with a nonlinear shear profile can be effectively represented by superposition of thin layers with linear shear, so this example gains certain relevance to the stability analysis of realistic physical flows.

2. Case of double transformation

Another interesting feature of the linear transformation mechanism we have found is the double transformation. Namely, for a certain choice of parameters $\varepsilon^2$ and $\sigma^2$ (looking at the dispersion curves Figs. 12 or 18) below we gather that there are two degeneration regions where $\Omega_1(\tau)$ and $\Omega_2(\tau)$ approach each other. In Figs. 7–11 and 13–17, we present numerical solutions of Eqs. (37)–(40) for different sets of physical parameters. It can be seen from the graphs that, in both cases, there is an energy exchange between oscillatory modes $\Omega_1(\tau)$ and $\Omega_2(\tau)$ in the DRs, where these two dispersion curves are coming close to each other. This exchange can be unambiguously seen from Figs. 7–11 and 13–17, marking drastic changes in the amplitudes of the oscillations of the physical quantities during their presence in both
DRs. However, these two cases differ significantly as each of them represents a different physical situation. In the first case, after passing through the second degeneracy region, the final oscillatory mode remains a SMW (Fig. 12) while in the second case, the final oscillatory mode is a FMW (Fig. 18). Although we stress, that in both cases initially, only the SMW mode was precisely excited. Explanation of this novel effect can be given by the following reasoning. After passing the first DR, a mixture of SMWs and FMWs will be present in the flow. In Ref. 23 it was demonstrated that the outcome of the amplification process of the slow magnetosonic wave in incompressible, magnetized Couette flow with isotropic thermal pressure significantly depends on the phase with which the wave enters the amplification region. The physical system under present consideration differs from the one studied in Ref. 23 (as in our case we have two oscillatory modes). But, presumably, the outcome of the process of energy exchange between SMWs and FMWs depends on the phase difference between these oscillatory modes at the time

FIG. 10. Numerical solutions of the Eqs. (37)–(40) for the set of the following physical parameters: \( \beta(0)=5, R=0.05, \sigma^2=1, \) and \( \epsilon^2=0.16. \) The solution was obtained with precise initial conditions \( D(0)=0.086 \), \( v(0)=0.100 \), \( v_i(0)=-0.002 \), \( 650 \), \( 656,2 \), and \( b(0)=0.002 \), \( 650 \), \( 656,2 \) so as to distinctly excite the \( \Omega_1(\tau) \) SMW mode. This set of figures (Figs. 7–11) represents the double transformation when an initially excited SMW, after passing two degeneracy regions remains, a SMW (see the details in the text).

FIG. 11. Numerical solutions of the Eqs. (37)–(40) for the set of the following physical parameters: \( \beta(0)=5, R=0.05, \sigma^2=1, \) and \( \epsilon^2=0.16. \) The solution was obtained with precise initial conditions \( D(0)=0.086 \), \( v(0)=0.100 \), \( v_i(0)=-0.002 \), \( 650 \), \( 656,2 \), and \( b(0)=0.002 \), \( 650 \), \( 656,2 \) so as to distinctly excite the \( \Omega_1(\tau) \) SMW mode. This set of figures (Figs. 7–11) represents the double transformation when an initially excited SMW, after passing two degeneracy regions remains, a SMW (see the details in the text).

FIG. 12. Here \( E_{\text{tot}}(\tau)/E_{\text{tot}}(0) \) (solid curve), \( \Omega_1(\tau)/\Omega(0) \) (i.e., SMW) [dashed curve, which overlaps on \( E_{\text{tot}}(\tau)/E_{\text{tot}}(0) \) in the time interval \( 0<\tau<70.66 \)], and \( \Omega_2(\tau)/\Omega(0) \) [dash–dotted curve] for the same set of physical parameters as in Figs. 7–11. Here \( \tau=70.66 \) is a midpoint of the DR. After passing the first DR, \( E_{\text{tot}} \) deviates from SMW as it gets an admixture of the FMW and, finally, after the second DR, it again almost exactly returns to a SMW. In other words, this graph represents the process SMW—the mixture of SMW and FMW—SMW.

FIG. 13. Numerical solutions of Eqs. (37)–(40) for the set of the following physical parameters: \( \beta(0)=5, R=0.05, \sigma^2=1, \) and \( \epsilon^2=0.2. \) The solution was obtained with precise initial conditions \( D(0)=0.084 \), \( v(0)=0.100 \), \( 155,13 \), \( v_i(0)=-0.003 \), \( 102 \), \( 600,6 \), and \( b(0)=0.003 \), \( 102 \), \( 600,6 \) so as to distinctly excite the \( \Omega_1(\tau) \) SMW mode. This set of figures (Figs. 13–17) represents the double transformation when an initially excited SMW, after passing two degeneracy regions, finally almost completely transforms into a FMW (see the details in the text).
moment when the mixture leaves the second DR. The latter
depends on the duration of the presence of the mixture in the
second DR (which, in turn, is determined by the R
parameter—the smaller the R the longer the time the mixture
remains in the DR) and on physical parameters $\epsilon^2$ and $\sigma^2$. So
far, for different sets of physical parameters (i.e., for an op-
posite phase difference at the instant when the mixture leaves
the second DR) we can end up either with SMWs or FMWs.

Besides this, it is worthwhile to comment on the behav-
ior of $E_{\text{tot}}$ in the latter two cases (Figs. 12 and 18), where we
plotted quantities $E_{\text{tot}}(\tau)/E_{\text{tot}}(0)$, $\Omega_1(\tau)/\Omega_1(0)$, and $[\Omega_2(\tau)/\Omega_2$
(midpoint of the first DR)×$\Omega_1$ (the midpoint of the first
DR)/$\Omega_1(0)$]. Again the latter quantity is rescaled so as to
reveal the true dispersion of the SMWs and FMWs. On both
graphs we see that, initially, $E_{\text{tot}}(\tau) \sim \Omega_1(\tau)$ as it should be,
since we excited the SMW. But, this lasts until the oscilla-
tory mode is far from the first DR (about $\tau=70.66$ in one
case and $\tau=71.71$ in the other). However, we clearly see that
after passing the first DR, $E_{\text{tot}}$ deviates from the SMW dis-
\begin{align*}
\begin{array}{c}
\text{FIG. 15. Numerical solutions of Eqs. (37)–(40) for the set of the following physical parameters: } \\
\beta(0)=5, R=0.05, \sigma^2=1, \text{ and } \epsilon^2=0.2. \text{ The solution was obtained with precise initial conditions } D(0)=0.084486997, \\
v_r(0)=0.10015513, v_f(0)=-0.0031026006, \text{ and } b(0)=0.0031026006 \text{ so as to distinctly excite the } \Omega_1(\tau) \text{ SMW mode. This set of figures (Figs. 13–17) represents the double transformation when an initially excited SMW, after passing two degeneracy regions, finally almost completely transforms into a FMW (see the details in the text).}
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\text{FIG. 16. Numerical solutions of Eqs. (37)–(40) for the set of the following physical parameters: } \\
\beta(0)=5, R=0.05, \sigma^2=1, \text{ and } \epsilon^2=0.2. \text{ The solution was obtained with precise initial conditions } D(0)=0.084486997, \\
v_r(0)=0.10015513, v_f(0)=-0.0031026006, \text{ and } b(0)=0.0031026006 \text{ so as to distinctly excite the } \Omega_1(\tau) \text{ SMW mode. This set of figures (Figs. 13–17) represents the double transformation when an initially excited SMW, after passing two degeneracy regions, finally almost completely transforms into a FMW (see the details in the text).}
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\text{FIG. 17. Numerical solutions of Eqs. (37)–(40) for the set of the following physical parameters: } \\
\beta(0)=5, R=0.05, \sigma^2=1, \text{ and } \epsilon^2=0.2. \text{ The solution was obtained with precise initial conditions } D(0)=0.084486997, \\
v_r(0)=0.10015513, v_f(0)=-0.0031026006, \text{ and } b(0)=0.0031026006 \text{ so as to distinctly excite the } \Omega_1(\tau) \text{ SMW mode. This set of figures (Figs. 13–17) represents the double transformation when an initially excited SMW, after passing two degeneracy regions, finally almost completely transforms into a FMW (see the details in the text).}
\end{array}
\end{align*}
there was an energy exchange between the SMW and the FMW. Further, from the first DR until the second DR there is a mixture of $\Omega_1(\tau)$ and $\Omega_2(\tau)$ propagating. But in one case (Fig. 12) after leaving the second DR we have again SMWs, whereas in another case (Fig. 18) we finally have FMWs.

**B. Unstable oscillatory modes**

In this section we study unstable oscillatory modes that can exist in the flow with anisotropic thermal pressure, namely, firehose and mirror instabilities and their interplay with the effect of the presence of the shear. First, let us derive the stability criteria in the flow under study. Equation (43) implies that the oscillatory mode becomes unstable when $\omega_1^2 \omega_2^2 \lambda_1^2(\tau) = e^4 \beta^2(\tau)$. Further, using expressions for $\omega_1$ and $\omega_2$ and also straightforward equalities $[\cos^2 \theta = 1/(1 + \beta^2(\tau))]$ and $[\sin^2 \theta = \beta^2(\tau)/(1 + \beta^2(\tau))]$ we arrive at the condition

$$\sin^2 \theta \cdot e^4 - 3(1 + \sin^2 \theta) \cdot e^2 + 3(\cos^2 \theta - \sigma^2) > 0,$$

(47)

and its fulfillment ensures the unstable behavior of the anisotropic flow. We can obtain the traditional criterion $^5^3$ (adopted for our notations, of course) for firehose instability by simply putting $\theta = 0$ in the latter formula, which yields $e^2 + \sigma^2 < 1$. Doing the same, but with $\theta = \pi/2$, we can obtain the criterion for the mirror instability, $\sigma^2 + 2e^2 < e^4/3$.

There are a variety of physically interesting cases that arise in this consideration. But, we restrict ourselves to two examples of the firehose instability and one example of the mirror instability, which in our opinion are the most interesting from the point of view of possible experimental implications.

Before starting consideration of the numerical results, we would like to emphasize a very important difference that arises due to the presence of the shear in the flow. If we look at the dispersion curves, Figs. 20 and 22, we can see that the unstable region for the firehose instability is located in the middle of the graphs, i.e., around $\tau = 100$, where the real part of the $\Omega_1(\tau)$ curve (the curve with circles), on which actually this instability occurs, remains at zero, while it also has a nonzero imaginary part, which, of course, is not drawn on the graphs. Now if there would be no shear, in the flow under consideration either the SMW or the FMW could propagate with the fixed frequency given by any point corresponding to $\Omega_1(\tau)$ or $\Omega_2(\tau)$, or their linear superposition. But, the point is, that with the presence of the shear a linear drift does exist in
FIG. 22. The $k$ space, which basically means that if we excite a wave, say, SMW with initial frequency $\Omega_1(0)$ (Figs. 20 or 22), it will evolve, changing its frequency following the $\Omega_1(\tau)$ curve. Then, at that moment in time when it reaches the unstable region, the firehose instability will switch on, leading to the growth of the wave. Whereas in the case without shear, we would have a stable wave propagating with its initial fundamental frequency $\Omega_1(0)$. So, we conclude that in the shear flow, the occurrence of the firehose instability on the SMW branch of dispersion is unavoidable (if the appropriate condition $\varepsilon^2 + \sigma^2 < 1$ is fulfilled, of course). Basically, the same conclusions apply to the mirror instability as well (see Fig. 24), but in contrast to the firehose instability, which has a finite instability region, the mirror instability has two infinite unstable regions $(-\infty; -\tau_{\text{inst}}]$ and $[\tau_{\text{inst}}; +\infty)$, where

$$\tau_{\text{inst}} = \frac{1}{K} \left( \beta(0) - \sqrt{\frac{\varepsilon^2 + \sigma^2 - 1}{(\varepsilon^2/3) - 2\varepsilon^2 - \sigma^2}} \right).$$

1. Case of firehose instability

Now, turning back to the firehose instability, we can make much more far-reaching conclusions. Suppose we excited a FMW, which is apparently stable in what is called “standard MHD theory” [since $\Omega_1(\tau)$ itself never becomes negative]. But, in the case when we have a shear in the flow and there is the possibility of transformation (i.e., energy exchange between SMW and FMW), FMW can gain a significant part of the energy from the unstable SMW branch. Further, of course, at some stage, nonlinear effects will switch on and in the end the flow will lose stability, i.e., become turbulent. Now we discuss concrete examples of our numerical work. In Figs. 19 and 21 we present numerical solutions of Eqs. (37)-(40) for different sets of physical parameters. In both cases FMWs initially were excited. Looking at the corresponding dispersion curves in Figs. 20 and 22, we see that in both cases there is an unstable region on the SMW. To avoid any doubt, we choose physical parameters in such a way that the growth rates [i.e., the imaginary part in $\Omega_1(\tau)$ of the firehose instability are the same in both cases [in other words, in the midpoint of the instability region ($\beta=0$) the quantity $\varepsilon^2 + \sigma^2 - 1$ is the same] and the $K$ parameter is also the same, thus ensuring the same duration of the FMW in the region where $\Omega_1(\tau)$ and $\Omega_2(\tau)$ approach each other. But in the first case, we see that the FMW gained a significant amount of energy from the unstable SMW, while in the second case there is no amplification of FMW at all. The explanation of this fact is clear. From the dispersion curves (Figs. 20 and 22) we gather that, in the first case, the dispersion curves $\Omega_1(\tau)$ and $\Omega_2(\tau)$ (curves with circles) come rather close to each other and thus create two DRs, while in the second case the curves pass by at a significant distance and hence the formation of the DR is avoided. We clearly see from Fig. 20 that although initially the FMW was excited, after passing the first DR the total energy of the wave increased significantly, since it gained energy from the firehose-unstable SMW, finally reaching values two orders of magnitude more than its initial value. In Fig. 22 we see that there is no exchange of energy between the FMW and SMW since we see that during the whole elapsed time interval, $E_{\text{tot}}$ is strictly proportional to $\Omega_2(\tau)$, which actually was excited right from the beginning. We would like to mention that this interesting effect can occur because of the existence of shear in the flow. This was revealed thanks to the nonmodal approach, while it was not perceived under the more traditional normal modes paradigm.

2. Case of mirror instability

We also studied the case when the mirror instability can occur in the flow. Our numerical results are presented in Figs. 23 and 24, where we plot numerical solutions of Eqs.
(37)–(40), the corresponding dispersion curves, and the total energy. As an illustrative example we again chose the case when FMWs were initially excited (since it is rather novel and unusual when we have the amplification of the "classically stable" FMWs). We see that although the dispersion curves of $\Omega_1(\tau)$ and $\Omega_2(\tau)$ do not approach each other, we still see the amplification of the FMW, i.e., energy exchange between FMWs and SMWs still occurs. This can be explained if we recall the definition [Eq. (34)] of the coupling constant in this case. In fact, the latter one is proportional to $\epsilon^2$. To satisfy the mirror instability condition we need to require large values of $\epsilon^2$, which in the considered case is 10. This means that in this case coupling between SMWs and FMWs is so strong that in spite of the fact that the dispersion curves do not come close enough to each other, energy exchange between the oscillatory modes still occurs, thus leading to the amplification of FMW. Therefore, we conclude, that if the condition for the occurrence of the mirror instability is satisfied, the amplification of the FMW seems to be unavoidable.

As we can see from Fig. 24, the total energy of FMW goes to large negative values. One should not be surprised by this fact as first, strictly speaking, the total energy, as we mentioned above, is not a positively defined quantity, since it contains negative terms, (but this is an ad hoc explanation); second, normally in plasma physics it is possible to have waves with negative energies\textsuperscript{54–57} which basically means, that to excite such a wave no net energy needs to be introduced into the medium. Contrary to that, energetically is more profitable when the wave gives away its energy to the medium. In other words, it can be said that the medium is nonequilibrium. In the case of the firehouse instability (Fig. 20), the $E_{\text{tot}}$ also falls to negative values while being in the unstable region (however, this is obscured in the graph). Basically, the FMW, after passing the first DR gets an admixture of SMWs, which has obviously negative energy (since it is unstable). However, the instability region has a finite width and the SMW further becomes stable, which therefore ensures that energy again becomes positive with much increased magnitudes. In the case of the mirror instability, the FMW gained an admixture of SMW, which remains unstable in the infinite time interval $[\tau_{\text{end}}+\infty)$ with a permanently increasing growth rate of instability. Thus, the total energy of the wave will be negative all the time.

**IV. CONCLUSION**

We have observed from the above study that pressure anisotropy brought significant novelty to the consideration through two key aspects. The first is the possibility of the creation of the two degeneration regions, which, in turn, yields a novel phenomenon that we have called the double transformation. This includes an interesting alternative in the final outcome of the transformation process when there is a possibility to end up with either a SMW or a FMW. The second is the possibility of the occurrence of the firehose and mirror instabilities and their interplay with the effect of the presence of the shear in the flow, including, the possibility of a remarkable amplification of the FMW caused by the energy exchange with the firehose-unstable SMW (when the appropriate conditions are fulfilled); or unavoidable amplification of the FMW (due to the strong coupling) caused by the energy exchange with mirror-unstable SMW. It must be stressed that all novel effects are caused purely by the presence of the shear in the flow (through the time dependence of the fundamental frequencies, which, in turn, is a consequence of the linear drift in k space). And, of course, obtaining the new results became possible thanks to the relevant description of a physical system such as shear flow—i.e., in the framework of the nonmodal approach.
Although examples considered in this paper are rather idealistic, they still may be useful for understanding the stability of plasma in realistic laboratory devices such as tokamaks or MHD generators (we made more detailed remarks concerning this point in relevant places in the text). Besides this, plasma instabilities also play an important role in the formulation of radiation mechanisms from various space objects. [For instance, in pulsar winds the shear effect may have importance beyond the coronation radius. In addition, bearing in mind a condition $\epsilon^2 = (C_L/C_0)^2 = P_L^2/P_0^2 < 1$, relevant for this outflow, there is the possibility of the fulfillment of the criterion of the firehose instability ($\epsilon^2 + \sigma^2 < 1$), which finally could result in an interplay between the phenomena similar to that which was considered in this paper.] Thus, presumably, some of the effects studied here may contribute to the understanding of some of the astrophysical problems as well. Primarily, the aim of this paper was to investigate these effects theoretically, in a rather simplified framework. However, if one should desire to be more precise and rigorous in the statements when it comes to realistic physical or astrophysical situations, then more complicated flow geometries and symmetries should be taken into consideration. But, in our opinion, this paper can serve as a basic starting point for this task.

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20Lord Kelvin (W. Thomson), Philos. Mag. 24, 5, 188 (1887).
52In other words, this condition, which is relevant in the case of weak coupling, implies that $\omega_0 = \omega_c$. The latter, in turn, implies that the maximum energy exchange between the pendulums (see below) occurs when they have approximately the same length.